THE COMBINATORIAL LAPLACIAN ON CAYLEY GRAPHS

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Abstract. The Analytic Laplacian is studied throughout mathematics and physics. However on certain domains it is difficult to compute its eigenfunctions and eigenvalues. In this project we explore ways to approximate the Analytic Laplacian with a finite dimensional operator–the combinatorial Laplacian. We also explore some of the properties of the combinatorial Laplacian on Tori.

Contents

1. Introduction

The analytic Laplacian is a differential operator on a surface, or more generally manifold, that is ubiquitous in mathematics and physics. It is defined as either the positive or negative sum of second derivatives, depending on preference, and naturally arises when one compares how a function on a compact surface deviates from the curvature of that surface. In fact, the eigenfunctions of the operator correspond to energy levels of the surface in the following sense: think of a vibrating surface as a function on the surface: the energy associated with this vibration is related to the eigenvalues of the Laplacian, and the profile of the vibration is encoded in the associated eigenfunction. Think of standing waves on a vibrating string. The profile of the wave is an eigenfunction of the Laplacian on a line segment, and the energy associated to the wave is connected to the eigenvalues. These observations led Kac to ask the now famous question: “Can you hear the shape of a drum?”. In other words, if you know the eigenvalues of the Laplacian, can you determine entirely, the shape of the vibrating membrane. It turns out that the answer is false in general, but it took mathematicians a very long time to find a counterexample.

There is also a connection between the analytic Laplacian and "straight lines", or geodesics, on the surface. Moreover, in some circumstances, it governs the behavior of dynamical systems on the surface (eg. it can describe rates at which points flowing along geodesics cover the surface).

Because of this, or perhaps, fundamental beneath this, the Laplacian is central in many mathematical fields: Fourier and harmonic analysis, representation theory of groups, dynamical systems, quantum chaos, and many more.

In most circumstances, it is very difficult, even impossible, to directly compute the eigenfunctions and eigenvalues of the Laplacian. In the present day, there are many methods and programs for approximating the eigenvalues, for example, using finite element analysis. In this paper, we will study methods for approximating these eigenvalues with an eye towards creating a method for understanding the operator on other surfaces, such as translation surfaces. Can one use a discrete operator to make conjectures about the properties of the eigenvalues?

The first step is to define the correct form of a discrete operator that will approximate the Laplacian. In Section 2, we define the analytic Laplacian. Motivated by the general properties of the analytic Laplacian, we define a discrete operator that shares an important property with the analytic Laplacian–a mean value property. This discrete operator is the graph Laplacian, an operator associated with a graph that we can think of as embedded on a surface. This is our first candidate for the discrete operator intended to approximate the analytic Laplacian. However, using elements of Fourier analysis on abelian groups, we observe that the graph Laplacian is a Cayley matrix (see Subsection 2.1 for a definition), which enables us to quickly compute the eigenvalues of the graph Laplacian, and we observe that they are bounded, no matter how many vertices the graph has. We conclude there is no way this could be the correct discrete operator: the eigenvalues of the analytic Laplacian are unbounded.

In Section 3, we observe that the issue seems to be related to the geometry of the circle. The embedded graphs that we use corresponding to the graph Laplacian do not observe lengths on the circle. The metric is a missing element: the analytic Laplacian does not exist on a surface unless you first define a metric, or way to measure distances, on the space. To do this, we rely on an older method due to Dodziuk in which he combines elements of algebraic topology and differential geometry and shows how to include the geometric information in the graph setting. We construct a combinatorial Laplacian with the correct properties by picking the correct inner product on a singular cochain. We work on the circle, and observe that as we increase the number of vertices in our graph, the eigenvalues of the combinatorial Laplacian do indeed converge to the eigenvalues of the analytic Laplacian.

In Section 4, we develop the machinery necessary for an explicit definition of the combinatorial Laplacian for the square torus, which is special example of a translation surface.

In Section 5, we make an observation: while the eigenvalues of the graph Laplacian and the combinatorial Laplacian differ, they share the same eigenspaces. We ask if this means that the eigenspaces for the Laplacian in the singular cochain are independent of the choice of inner product. The way we show this is closely tied to a collection of isometries, which happen to coincide with fundamental group.

In Section 6, we discuss possible future work.

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2. Analytic and graph laplacian

Definition 1. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable. We define the Laplacian of f as*

$$\Delta f = -div(grad(f)) = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \tag{1}$$

An important class of functions are the harmonic functions. A harmonic function on $\mathbb{R}^n$ is a function which solves the differential equation

$$\Delta f = 0$$

One very nice property of harmonic functions is the "mean value property."

Remark 1. *Suppose $f : \mathbb{C} \to \mathbb{R}$ is harmonic on the set $\Omega \subseteq \mathbb{C}$. Then, for any $z_0 \in \Omega$ we have*

$$f(z_0) = \frac{1}{2\pi r} \int_{\partial D(z_0,r)} f(z)dz$$

for any $r < d(z_0, \partial\Omega)$.

In algebraic topology, manifolds are often studied using triangulations. By triangulating a manifold it is possible to determine information about the equivalence class in which the manifold belongs. One interesting question is whether or not we can generalize the idea of harmonic functions to graphs. Every triangulation of a manifold can be thought of as a graph, so in describing the harmonic functions on the vertices of the triangulation we might be able to define a finite dimensional analogue of the Laplacian.

We begin by noting that our harmonic functions should satisfy a property similar to Remark 1. This is the motivation for the following definition.

Definition 2. *Let $G = (V, E)$ be a simple, connected graph. We say a function $f : V \to \mathbb{R}$ is harmonic if*

$$f(v) = \frac{1}{d(v)} \sum_{w \sim v} f(w)$$

Where $w \sim v$ if and only if $\{w, v\} \in E$.

We want to define a new operator $\Delta_G : L^2(V) \to L^2(V)$ that satisfies

$$\Delta_G f = 0 \iff f \text{ is harmonic}$$

Definition 3. *For a simple graph G on vertices $\{v_1, \ldots, v_n\}$, we define the graph Laplacian*

$$\Delta_G = D_G - A_G$$

where D_G is a diagonal matrix whose i_{th} entry corresponds to the degree of the vertex v_i.

Note that if f is harmonic

$$\Delta_G f(v) = D_G f(v) - A_G f(v) = d(v)f(v) - \sum_{w\sim v} f(w) = 0$$

So this definition seems reasonable. It is still unclear whether this graph Laplacian actually approximates the analytic Laplacian though. If it does then we should see

1) The eigenvalues of the graph Laplacian converge to the eigenvalues of the analytic Laplacian as we increase the number of vertices in our graph
2) The eigenvectors of the graph Laplacian are obtained by sampling the eigenfunctions of the analytic Laplacian at every vertex of the triangulation

We will start by looking at the graph Laplacians obtained by embedding Cycle graphs in the circle. To make this easier we introduce a result from Fourier analysis on Abelian groups.

2.1. **Cayley Graphs.** In general, not much can be said about the structure of a graph. However, certain graphs possess the structure of a group. These are called Cayley graphs.

Definition 4. *Let $\mathbb{G} = \{g_1, \dots, g_n\}$ be a finite group and suppose S is an inverse closed subset of $\mathbb{G}$. Then we can define a graph on the elements of $\mathbb{G}$ such that $x \sim y$ if $x^{-1}y \in S$. We call this graph the Cayley graph of $\mathbb{G}$ with generating set S.*

Some authors do not impose the requirement that S be inverse closed. However, in the context of constructing a triangulation, it is important that we end up with an undirected graph.

Definition 5. *Let $\mathbb{G} = \{g_1, \dots, g_n\}$ be a finite group. Suppose $f : \mathbb{G} \to \mathbb{R}$ is an inverse invariant function. The Cayley matrix of $\mathbb{G}$ with respect to f is the matrix*

$$M_{\mathbb{G},f} := \begin{pmatrix} f(g_1^{-1}g_1) & \dots & f(g_1^{-1}g_n) \\ \vdots & \ddots & \vdots \\ f(g_n^{-1}g_1) & \dots & f(g_n^{-1}g_n) \end{pmatrix} \tag{2}$$

The i, j entry of $M_{\mathbb{G},f}$ is given by $f(g_i^{-1}g_j)$.

To illustrate this definition with a simple example, take $G = \mathbb{Z}_5$ and $f = \mathbb{1}_{\{1,4\}}$. Then, $M_{G,f}$ is just the adjacency matrix of the Cayley graph of $\mathbb{Z}_5$ with generating set $\{1, 4\}$.

Definition 6. *Let $\mathbb{G}$ be a compact group. A character of $\mathbb{G}$ is a continuous group homomorphism $\chi : \mathbb{G} \to \mathbb{T}$.*

Note that in the following lemma, we identify functions on $\mathbb{G}$ with vectors in $\mathbb{C}^n$.

Lemma 1. *Let $\mathbb{G} = \{g_1, \dots, g_n\}$ be an Abelian group. Suppose $f : \mathbb{G} \to \mathbb{R}$ is an inverse invariant function. Let $M_{\mathbb{G},f}$ be the Cayley matrix of $\mathbb{G}$ with respect to f. Then, the eigenvectors of $\mathbb{G}$ are the group characters $\widehat{\mathbb{G}}$. Moreover, for a given character $\chi \in \widehat{\mathbb{G}}$, the corresponding eigenvalue is given by*

$$\lambda_\chi = \sum_{g\in\mathbb{G}} f(g)\chi(g)$$

Proof. Let $\chi \in \widehat{\mathbb{G}}$ be given. Let $i \in \{1, \dots, b\}$ be fixed. Note that

$$(M_{\mathbb{G},f}\chi)_i = \sum_{k=1}^{n} f(g_i^{-1}g_k)\chi(g_k)$$

Since $\mathbb{G}$ is a finite group we can sum over the elements in $\mathbb{G}$ in any order. By performing a "change of variables" $g_k \mapsto g_k g_i$ we obtain

$$\begin{aligned}(M_{\mathbb{G},f}\chi)_i &= \sum_{k=1}^{n} f(g_i g_k g_i^{-1})\chi(g_k g_i)\\ &= \left(\sum_{k=1}^{n} f(g_k)\chi(g_k)\right)\chi(g_i)\\ (M_{\mathbb{G},f}\chi)_i &= \left(\sum_{g\in\mathbb{G}} f(g)\chi(g)\right)\chi(g_i)\end{aligned}$$

Note that the constant by which $\chi(g_i)$ is scalled is independent of i. Therefore,

$$M_{\mathbb{G},f}\chi = \left(\sum_{g\in\mathbb{G}} f(g)\chi(g)\right)\chi$$

Note that in the above equation we see

$$\lambda_\chi = \sum_{g\in\mathbb{G}} f(g)\chi(g)$$

□

To more concretely illustrate this result, consider the group $\mathbb{Z}_5$ with generating set $S = \{1, 4\}$. Now, the Cayley graph of this group (figure 1) is just the cycle graph on 5 vertices. Moreover, the adjacency matrix of this graph is a Cayley matrix of $\mathbb{Z}_5$ corresponding to the function $f = \mathbb{1}_S$. The dual group of $\mathbb{Z}_5$ is given by

$$\widehat{\mathbb{Z}_5} = \{\chi_0, \chi_1, \ldots, \chi_4\}$$

where for $k = 0, 1, \ldots, 4$ and $n \in \mathbb{Z}_5$ we have

$$\chi_k(n) = \exp\left(\frac{2\pi i k n}{5}\right)$$

Let $\omega = \exp(2\pi i/5)$. By thinking of each χ_k as a vector we have

$$\chi_0 = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \chi_1 = \begin{pmatrix}1\\\omega\\\omega^2\\\omega^3\\\omega^4\end{pmatrix}, \chi_2 = \begin{pmatrix}1\\\omega^2\\\omega^4\\\omega^1\\\omega^3\end{pmatrix}, \chi_3 = \begin{pmatrix}1\\\omega^3\\\omega^1\\\omega^4\\\omega^2\end{pmatrix}, \text{and } \chi_4 = \begin{pmatrix}1\\\omega^4\\\omega^3\\\omega^2\\\omega^1\end{pmatrix}$$

Now, identify the eigenvalue λ_k with the eigenvector χ_k. It is clear that $\lambda_0 = 2$, since G is a two regular graph. Applying lemma 1 we have

$$\lambda_k = \chi_k(1) + \chi_k(-1) = 2\cos\left(\frac{2\pi k}{5}\right)$$

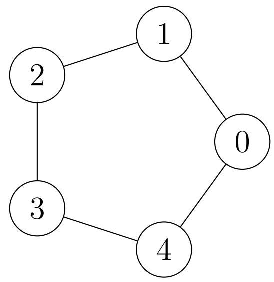

FIGURE 1. Cayley Graph of $\mathbb{Z}_5$ with generating set $\{1, 4\}$

2.2. **The Spectrum of the Graph Laplacian on the Cycle Graphs.** We begin by noting that the spectrum of the analytic Laplacian on $\mathbb{T}$ is the set $\{4\pi^2 n^2\}_{n=0}^{\infty}$. If the graph Laplacian is capable of approximating the analytic Laplacian then we should see the spectrum of Δ_{G_n} converge to the spectrum of Δ as we increase the vertices in our triangulation $K_n = V(G_n) \cup E(G_n)$.

For every $n \in \mathbb{N}$ let $G_n = \mathrm{Cay}(\mathbb{Z}_n, \{1, n-1\})$. Recall from Definition 3 that

$$\Delta_{G_n} = D_{G_n} - A_{G_n}$$

Since G_n is a 2-regular graph, this simplifies to

$$\Delta_{G_n} = 2I - A_{G_n}$$

You might notice that we can construct an inverse invariant function $f : \mathbb{Z}_n \to \mathbb{R}$ such that $\Delta_{G_n} = M_{\mathbb{Z}_n, f}$. Or you might notice that for $k = 0, \ldots n-1$ we have

$$\Delta_{G_n}\chi_k = 2\chi_k - A_{G_n}\chi_k = \left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right)\chi_k$$

In either case, it is clear that the spectrum of Δ_{G_n} is bounded above by 4 and bounded below by 0. It appears that the graph Laplacian is not approximating the analytic Laplacian in the way that we would hope.

To understand why this is happening we need to take a step back. Our goal is to create a finite dimensional analogue of the Laplacian defined on twice differentiable functions on a manifold. This manifold has geometric properties that our graph does not. For instance, we could embed the same graph two circles of different lengths, but the graph Laplacian would appear to be the same in both cases.

3. Another Perspective on the Laplacian

In differential topology, manifolds are studied using the De Rham complex. Given a smooth manifold X of dimension N, the De Rham complex is the cochain complex

$$0 \xrightarrow{d} C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2) \xrightarrow{d} \ldots \xrightarrow{d} C^\infty(\Lambda^N) \xrightarrow{d} 0$$

Where $C^\infty(\Lambda^q)$ consists of the C^∞ q-forms on X and d is the exterior derivative. For any $q \in \{0, \ldots, N\}$ we can equip $L^2(\Lambda^q)$ with the inner product

$$\langle f, g\rangle_{L^2(\Lambda^q)} = \int_X f \wedge *g \tag{3}$$

In this context, the Analytic Laplacian is defined as

$$\Delta = dd^* + d^*d \tag{4}$$

where d^* is the adjoint of the differential with respect to the inner product defined in 3. If we triangulate X with a triangulation K, we can construct a real-chain complex

$$0 \xrightarrow{\partial} C_N(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1(K) \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} 0$$

One important thing to note is that for any $q \in \{0, \dots, N\}$, the vector space $\Lambda^q(X)$ is infinite dimensional while $C_q(K)$ is a finite dimensional vector space. By dualizing the chain complex we obtain the co-chain complex

$$0 \xrightarrow{\delta} C^0(K) \xrightarrow{\delta} C^1(K) \xrightarrow{\delta} \dots \xrightarrow{\delta} C^N(K)$$

Jumping ahead slightly, the combinatorial laplacian on the q_{th} cochain group is defined by

$$\Delta_q = \delta\delta^* + \delta^*\delta \tag{5}$$

where δ^* denotes the adjoint of δ. In our work we will focus on Δ_0 in particular. It was proven by Dodziuk in [1] that the C^∞ q-forms on X can be approximated by elements of $C^q(K)$ in the L^2 sense by subdividing K in an appropriate way.

3.1. **The Whitney Map.** There is one issue with the Combinatorial Laplacian as it is currently defined (5). We have not defined any inner product on $C^0(K, \mathbb{R})$ or $C^1(K, \mathbb{R})$. Without an inner product, we cannot define δ^*. In [3], Whitney proves that it is possible to pull back the inner product from $C^\infty(\Lambda^q)$ onto $C^q(K, \mathbb{R})$.

We first order the vertices of K. That is we label the vertices of K by $\{p_0\}, \dots, \{p_n\}$ such that $\{p_i\} \preceq \{p_j\}$ whenever $i \leq j$. The significance being, if we have an edge $\{p_i, p_j\}$ where $i \leq j$ then

$$\partial_2(\{p_i, p_j\}) = \{p_j\} - \{p_i\}$$

Now, identifying each chain $A \in C^q(K)$ with the linear combination

$$A = \sum_{\tau \in C_q(K)} a_\tau \tilde{\tau} \text{ where } a_\tau \in \mathbb{R}$$

where each τ is a $q - simplex$ $\{p_0, \dots, p_q\}$ of increasing vertices and $\tilde{\tau} : C_q(K) \to \mathbb{R}$ by

$$\tilde{\tau}(\sigma) = \begin{cases} 1 & \tau = \sigma \\ 0 & \text{otherwise} \end{cases}$$

Definition 7. [3] *Let $\tau = \{p_0, \dots, p_q\} \in C_q(K)$. Define $W\tilde{\tau} \in L^2(\Lambda^q)$ by*

$$W\tilde{\tau} = q! \sum_{k=0}^{q} (-1)^k \mu_{p_k} d\mu_{p_0} \wedge d\mu_{p_1} \wedge \dots \wedge d\mu_{p_{k-1}} \wedge d\mu_{p_{k+1}} \wedge \dots \wedge d\mu_{p_q}$$

where μ_x denotes the x_{th} barycentric coordinate in K.

It is noted by Dodziuk [1] that the the barycentric coordinates μ_p are not differentiable functions on X. An example of this can be seen in section (3.2). However, we can just work in $L^2(\Lambda^q)$; A larger function space which includes non-differential functions.

The following remark helps get a better sense of W.

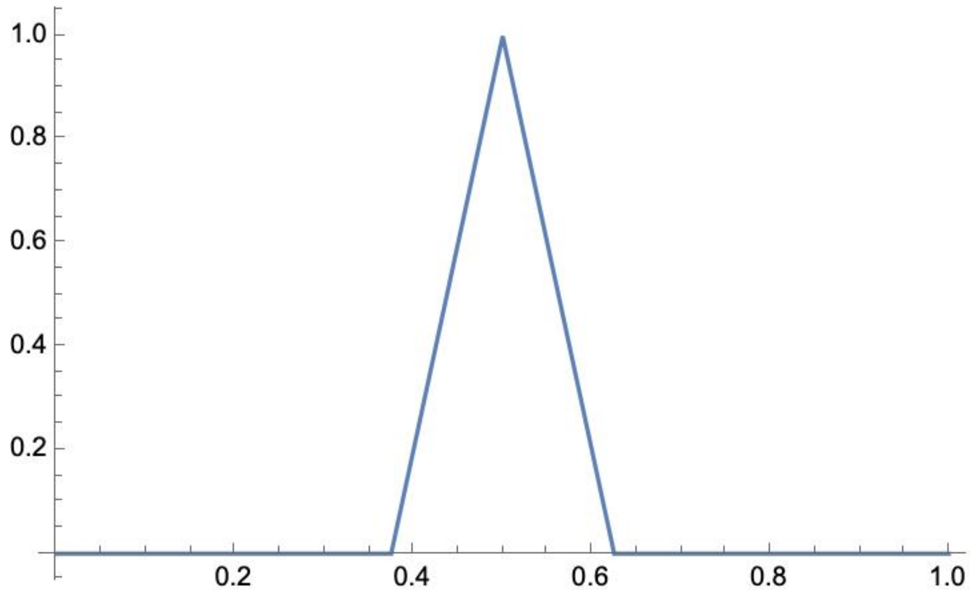

FIGURE 2. μ_4 when $n = 8$

Remark 2. [3] *Let $\tau \in C_q(K)$ be given. Let S_τ be the open star of τ. Then, $W\tilde{\tau}$ is supported on $\overline{S_\tau}$.*

Definition 8. [3] *Let $x, y \in C^q(K, \mathbb{R})$ be given. Define*

$$\langle x, y\rangle_{C^q(K)} = \langle Wx, Wy\rangle_{L^2(\Lambda^q)}$$

3.2. **Example on the Circle.** Let $X = \mathbb{T}$. Fix $n \in \mathbb{N}$. We define the triangulation $K_n = \{V, E\}$ where V, E are given by the graph $\mathrm{Cay}(\mathbb{Z}_n, \{1, n-1\})$. Define

$$\mathcal{B} := \{\{x, y\} : -x + y \in \{1, n-1\} \text{ and } x \leq y\}$$

and

$$\mathcal{C} = \{\{i\} : i \in \mathbb{Z}_n\}$$

We denote the vertex $\{i\}$ by v_i Our goal is to compute Δ_0. It is best to start with understanding what W does to elements of $C^0(K)$ and $C^1(K)$. Let $i \in \mathbb{Z}_n$ be given. Note that

$$W(\tilde{v_i}) = \mu_i \tag{6}$$

On the other hand,

$$W(\tilde{e_i}) = \mu_i d\mu_{i+1} - \mu_{i+1} d\mu_i \tag{7}$$

By thinking of the circle as the interval $[0, 1)$ after identifying 0 and 1, we define the barycentric coordinates of a point $x \in [0, 1)$ by

$$\mu_i(x) = 1 - n\left|x - \frac{i}{n}\right|$$

See Figure (2) for an example of such a function.

Lemma 2. *Suppose $i, j \in \mathbb{Z}_n$. Then,*

$$\langle \tilde{v_i}, \tilde{v_j} \rangle_{C^0(K)} = \begin{cases} \frac{2}{3n} \text{ if } j = i \\ \frac{1}{6n} \text{ if } j = i \pm 1 \\ 0 \text{ otherwise} \end{cases}$$

Proof. First observe that

$$\mu_i(x) = \mu_0\left(x - \frac{i}{n}\right)$$

So, $\langle \mu_i, \mu_i \rangle = \langle \mu_0, \mu_0 \rangle$ for every $i \in \mathbb{Z}_n$. By definition of the inner product on $C^0(K)$ we have

$$\begin{aligned} \langle \tilde{v_i}, \tilde{v_i} \rangle &= \int_X \mu_0 \wedge *\mu_0 \\ &= \int_0^1 \mu_0(x)^2 dx \\ &= \int_0^1 \mu_0(x) \\ &= \int_{1-1/n}^{1/n} \mu_0(x)^2 dx \\ &= 2\int_0^{1/n} (1-nx)^2 dx \\ \langle \tilde{v_i}, \tilde{v_i} \rangle &= \frac{2}{3n} \end{aligned}$$

On the other hand, if $j = i - 1$ then the function $\mu_i \mu_j$ is supported on the interval $(\frac{i-1}{n}, \frac{i}{n})$. The inner product becomes

$$\langle \tilde{v_i}, \tilde{v_j} \rangle = \int_{(i-1)/n}^{i/n} (1 - n\left(\frac{i}{n} - x\right))(1 - n\left(x - \frac{i-1}{n}\right))dx = \frac{1}{6n}$$

The case for $j = i + 1$ is equivalent. Outside of these three cases, $\mu_i \mu_j$ is 0 everywhere on $[0, 1)$ and so any other inner product must be 0. ◻

In the proof of the following lemma, note that P denotes an $n \times n$ "forward shift" matrix. That is, $(Px)_i = x_{\overline{i+1}}$

Lemma 3. *Suppose $i, j \in \mathbb{Z}_n$. Then,*

$$\langle \tilde{e_i}, \tilde{e_j} \rangle = \begin{cases} n & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Proof. First recall that

$$W\tilde{e_i} = \mu_i d\mu_{i+1} - \mu_{i+1} d\mu_i$$

Moreover, $\mu_i \mu_{i+1}$ is supported on $[\frac{i}{n}, \frac{i+1}{n}]$ so on that interval we have

$$d\mu_i = -ndx \;\text{ and }\; d\mu_{i+1} = ndx$$

Hence,

$$W\tilde{e_i} = \mu_i(ndx) - \mu_{i+1}(-ndx) = (\mu_i + \mu_{i+1})(ndx) = ndx$$

Suppose $i = j$.

$$\langle \tilde{e}_i, \tilde{e}_j \rangle = \int_{S^1} W\tilde{e}_i \wedge *W\tilde{e}_j = \int_{i/n}^{(i+1)/n} ndx \wedge *ndx = \int_{i/n}^{(i+1)/n} n^2 dx = n$$

On the other hand, if $i \neq j$, then $W\tilde{e}_i$ and $W\tilde{e}_j$ have disjoint supports. Meaning that integrating $W\tilde{e}_i \wedge *W\tilde{e}_j$ over S^1 gives 0. □

Note that

$$\partial_0(v_i) = -v_i + v_{\overline{i+1}}$$

where $\overline{i+1} \equiv i+1 \pmod n$. Hence, we can construct the matrix of ∂_0 to find

$$M_{\mathcal{B},\mathcal{C}}(\partial_0) = P - I$$

It follows that

$$M_{\mathcal{C}',\mathcal{B}'}(\delta_0) = M_{\mathcal{B},\mathcal{C}}(\partial_0)^t = P^{n-1} - I$$

since δ_0 is the dual map of ∂_0 [2]. We can use this matrix to describe the action of δ_1 on the basis elements of $C^0(K)$. Importantly,

$$\delta_1(\tilde{v}_i) = \tilde{e}_{\overline{i-1}} - \tilde{e}_i \tag{8}$$

Lemma 4. *Let $i, j \in \mathbb{Z}_n$ be given. Then*

$$\langle \delta_0 \tilde{v}_i, \tilde{e}_j \rangle = \begin{cases} n & \text{if } j = i-1 \\ -n & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Proof. By (8) we have

$$\langle \delta \tilde{v}_i, \tilde{e}_j \rangle = \langle \tilde{e}_{\overline{i-1}}, \tilde{e}_j \rangle - \langle \tilde{e}_i, \tilde{e}_j \rangle$$

Note that if $j = i - 1$ then we have

$$\langle \tilde{e}_{\overline{i-1}}, \tilde{e}_j \rangle - \langle \tilde{e}_i, \tilde{e}_j \rangle = n - 0 = n$$

Moreover, if $j = i$ then we have

$$\langle \tilde{e}_{\overline{i-1}}, \tilde{e}_j \rangle - \langle \tilde{e}_i, \tilde{e}_j \rangle = 0 - n = -n$$

In any other case,

$$\langle \tilde{e}_{\overline{i-1}}, \tilde{e}_j \rangle - \langle \tilde{e}_i, \tilde{e}_j \rangle = 0 - 0 = 0$$

□

We now formulate the process through which we will compute δ_0^*. For each $j = 0, 1, \ldots, n-1$ we can set up the linear equation

$$\langle \tilde{v}_i, \delta_0^* \tilde{e}_j \rangle = \sum_{k=0}^{n} a_{j,k} \langle \tilde{v}_i, \tilde{v}_k \rangle$$

After evaluating the inner products, we obtain a linear system in terms of the coefficients $\{a_{j,k}\}_{j,k\in\mathbb{Z}_n}$. More concretely, we find that

$$\left(\frac{2}{3n} I + \frac{1}{6n} M_{\mathbb{Z}_n, \mathbb{1}_{\{1,n-1\}}} \right) [a_{j,k}]_{j,k\in\mathbb{Z}_n} = n\,(P - I)$$

Now, assuming that the lefthand matrix is invertible we have

$$[a_{j,k}]_{j,k\in\mathbb{Z}_n} = n^2\left(\frac{2}{3}I + \frac{1}{6}M_{\mathbb{Z}_n,\mathbb{1}_{\{1,n-1\}}}\right)^{-1}(P-I) \tag{9}$$

Lemma 5.

$$M_{\mathcal{C}',\mathcal{C}'}(\Delta_0) = n^2\left(\frac{2}{3}I + \frac{1}{6}M_{\mathbb{Z}_n,\mathbb{1}_{\{1,n-1\}}}\right)^{-1}L_{\mathbb{Z}_n} \tag{10}$$

Proof. Note that

$$M_{\mathcal{C}',\mathcal{B}'}(\delta_0) = P^{n-1} - I$$

and so applying δ_0^* to δ_0 we get

$$\begin{aligned}
M_{\mathcal{C}',\mathcal{C}'}(\Delta_0) &= M_{\mathcal{B}',\mathcal{C}'}(\delta_0^*)M_{\mathcal{C}',\mathcal{B}'}(\delta_0)\\
&= n^2\left(\frac{2}{3}I + \frac{1}{6}M_{\mathbb{Z}_n,\mathbb{1}_{\{1,n-1\}}}\right)^{-1}(P-I)(P^{n-1}-I)\\
M_{\mathcal{C}',\mathcal{C}'}(\Delta_0) &= n^2\left(\frac{2}{3}I + \frac{1}{6}M_{\mathbb{Z}_n,\mathbb{1}_{\{1,n-1\}}}\right)^{-1}L_{\mathbb{Z}_n}
\end{aligned}$$

□

Theorem 1. *The discrete Fourier transform diagonalizes Δ_0.*

Proof. First observe that taking

$$f = \frac{2}{3}\mathbb{1}_{\{0\}} + \frac{1}{6}\mathbb{1}_{\{1,n-1\}}$$

we have

$$M_{\mathbb{Z}_n,f} = \frac{2}{3}I + \frac{1}{6}M_{\mathbb{Z}_n,\mathbb{1}_{\{1,n-1\}}}$$

Hence, by lemma (1) the characters of $\mathbb{Z}_n$ are eigenvectors of $M_{\mathbb{Z}_n,f}$. So the unitary matrix

$$U_n = [\chi_i(j)/\sqrt{n}]_{i,j\in\mathbb{Z}_n} \tag{11}$$

diagonalizes $M_{\mathbb{Z}_n,f}$. In particular

$$M_{\mathbb{Z}_n,f} = U_n D_M U_n^*$$

where the k_{th} entry along D_M is

$$\lambda_{\chi_k} = \frac{2}{3} + \frac{1}{3}\cos\left(\frac{2\pi k}{n}\right)$$

Moreover, since $L_{\mathbb{Z}_n}$ is also a Cayley matrix corresponding to the function $g = 2\mathbb{1}_{\{0\}} - \mathbb{1}_{\{1,n-1\}}$ we have

$$L_{\mathbb{Z}_n} = U_n D_L U_n^*$$

where D_L is defined similarly to D_M. Now, we see that

$$\begin{aligned}
M_{\mathcal{C}',\mathcal{C}'}(\Delta_0) &= n^2(M_{\mathbb{G},f}^{-1})L_{\mathbb{Z}_n}\\
&= n^2(U_n D_M U_n^*)^{-1}U_n D_L U_n^*\\
&= n^2 U_n D_M^{-1} U_n^* U_n D_L U_n^*\\
M_{\mathcal{C}',\mathcal{C}'}(\Delta_0) &= n^2 U_n D_M^{-1} D_L U_n^*
\end{aligned}$$

Since both D_M^{-1} and D_L are diagonal, we have that U_n diagonalizes Δ_0. □

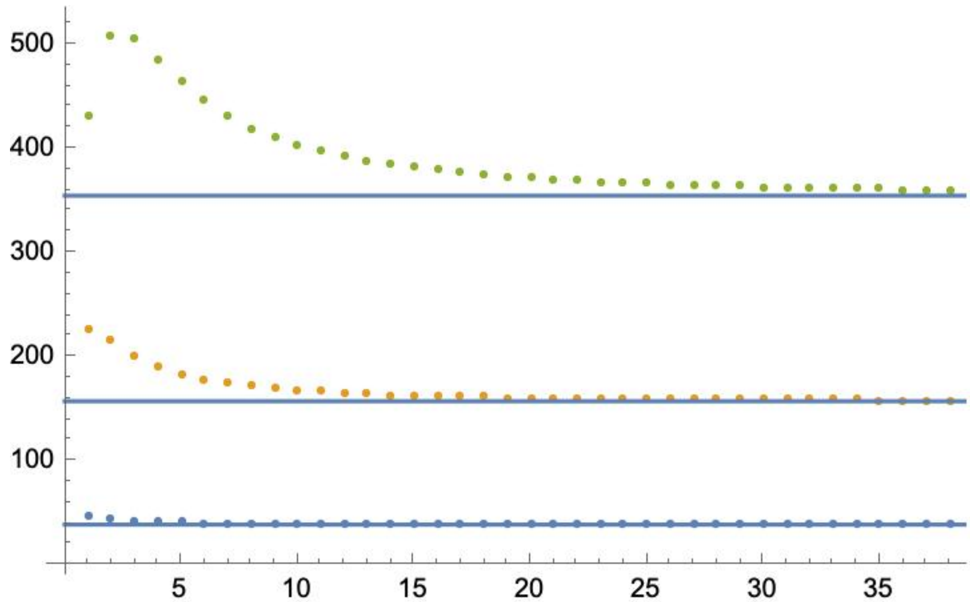

FIGURE 3. Eigenvalue Convergence of combinatorial laplacian to $4\pi^2$, $16\pi^2$, and $36\pi^2$

Corollary 1. *The eigenvectors of Δ_0 are the characters of $\mathbb{Z}_n$. The eigenvalue corresponding to χ_k (and χ_{-k}) is*

$$\lambda_k = n^2\left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right)\left(\frac{2}{3} + \frac{1}{3}\cos\left(\frac{2\pi k}{n}\right)\right)^{-1}$$

Proof. Since the matrix of Δ_0 is diagonalized by U_n it follows that each χ_k is an eigenvector of Δ_0.

Look at the k_{th} diagonal entry in $D_M^{-1}D_L$. □

3.3. **Computational results.** In [1], Dodziuk proves that the k_{th} eigenvalue of the combinatorial laplacian on $C^0(K)$ bounds the k_{th} eigenvalue of the analytic Laplacian in a desireable way. For any $k \le n$, there is some constant $C_k > 0$ such that

$$\lambda_k^n - C_k\eta_n \le \lambda_k \le \lambda_k^n$$

whenever $k \le d(n)$. In general η_n is a constant related to the size of the mesh which is approximating your surface. However, in our case, $\eta_n = 1/n$. The k_{th} eigenvalue of the analytic Laplacian is given by

$$\lambda_k = 4k^2\pi^2$$

To (computationally) verify that the combinatorial Laplacian on $C^0(\mathbb{Z}_n, \mathbb{R})$ is correct, we plot the Eigenvalues λ_1^n, λ_2^n, and λ_3^n of Δ_0 against the corresponding eigenvalues of the Analytic Laplacian (Figure 3).

4. Setup for Calculating the Combinatorial Laplacian on the Torus

Let $n \in \mathbb{N}$ be given. Define $S_n = \{(1,0), (n-1,0), (0,1), (0,n-1), (1,n-1), (n-1,1)\}$ Let $G_n = \text{Cay}(\mathbb{Z}_n \times \mathbb{Z}_n, S_n)$. Define $K_n = V_n \cup E_n \cup F_n$ where V_n and E_n are given by G_n and F_n is the set of faces in G_n.

4.1. **Barycentric Coordinate Functions.** We associate the Torus with $[0,1]^2$ in $\mathbb{R}^2$ by associating 0 with 1. Since the graph we embed inside the Torus is a Cayley graph it is sufficient to define the barycentric coordinate function corresponding to the basis element $\tilde{v}_{(0,0)}$. This will in turn define the barycentric coordinates for any $\tilde{v}_{(i,j)}$, where $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$ is arbitrary. This idea is more deeply explored in section 5.

Recall that in defining the barycentric coordinate functions on the circle, each coordinate function was a piecewise linear function uniquely defined by its value at a vertex and it's neighbors. The coordinate functions on the torus are defined similarly. Each coordinate function is piecewise planar supported on six triangular regions in $[0,1]^2$.

Let $n \in \mathbb{N}$ be fixed. The table below describes each triangular region.

Region	corner 1	corner 2	corner 3
$\mathcal{T}_1$	$(0,0)$	$(-1/n,0)$	$(0,-1/n)$
$\mathcal{T}_2$	$(0,0)$	$(0,-1/n)$	$(1/n,-1/n)$
$\mathcal{T}_3$	$(0,0)$	$(1/n,-1/n)$	$(1/n,-1/n)$
$\mathcal{T}_4$	$(0,0)$	$(1/n,0)$	$(0,1/n)$
$\mathcal{T}_5$	$(0,0)$	$(0,1/n)$	$(1/n,1/n)$
$\mathcal{T}_6$	$(0,0)$	$(-1/n,1/n)$	$(1-/n,0)$

Lemma 6. *Let $(x,y) \in [0,1]^2$ be given. Then,*

$$\mu_{(0,0)}(x,y) = \begin{cases} 1+nx+ny & \textit{if } (x,y) \in \mathcal{T}_1 \\ 1+ny & \textit{if } (x,y) \in \mathcal{T}_2 \\ 1-nx & \textit{if } (x,y) \in \mathcal{T}_3 \\ 1-nx-ny & \textit{if } (x,y) \in \mathcal{T}_4 \\ 1-ny & \textit{if } (x,y) \in \mathcal{T}_5 \\ 1+nx & \textit{if } (x,y) \in \mathcal{T}_6 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. For each triangular region, find the plane going the points $\mu_{(0,0)}((0,0))$, $\mu_{(0,0)}(p_1)$, and $\mu_{(0,0)}(p_2)$ where p_1 and p_2 are the other corners of the triangle. This determines the value of the barycentric coordinate function on each triangular region. ☐

We include an image of the function $\mu_{(0,0)}$ in Figure (4).

Lemma 7. *Let $(i,j) \in \mathbb{Z}_n \times \mathbb{Z}_n$ be given. Then*

$$\mu_{(i,j)}(x,y) = \mu_{(0,0)}\left(x - \frac{i}{n}, y - \frac{j}{n}\right) = \begin{cases} 1+n\left(x-\frac{i}{n}\right)+n\left(y-\frac{j}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_1 \\ 1+n\left(y-\frac{j}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_2 \\ 1-n\left(x-\frac{i}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_3 \\ 1-n\left(x-\frac{i}{n}\right)-n\left(y-\frac{j}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_4 \\ 1-n\left(y-\frac{j}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_5 \\ 1+n\left(x-\frac{i}{n}\right) & \textit{if } \left(x-\frac{i}{n}, y-\frac{j}{n}\right) \in \mathcal{T}_6 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Since we are working with the unit Torus, G_n is embedded such that for any point $g \in G_n$ we have

$$D(g, g+(0,1)) = D(g, g+(0,n-1)) = D(g, g+(1,0)) = D(g, g+(n-1,0)) = \frac{1}{n}$$

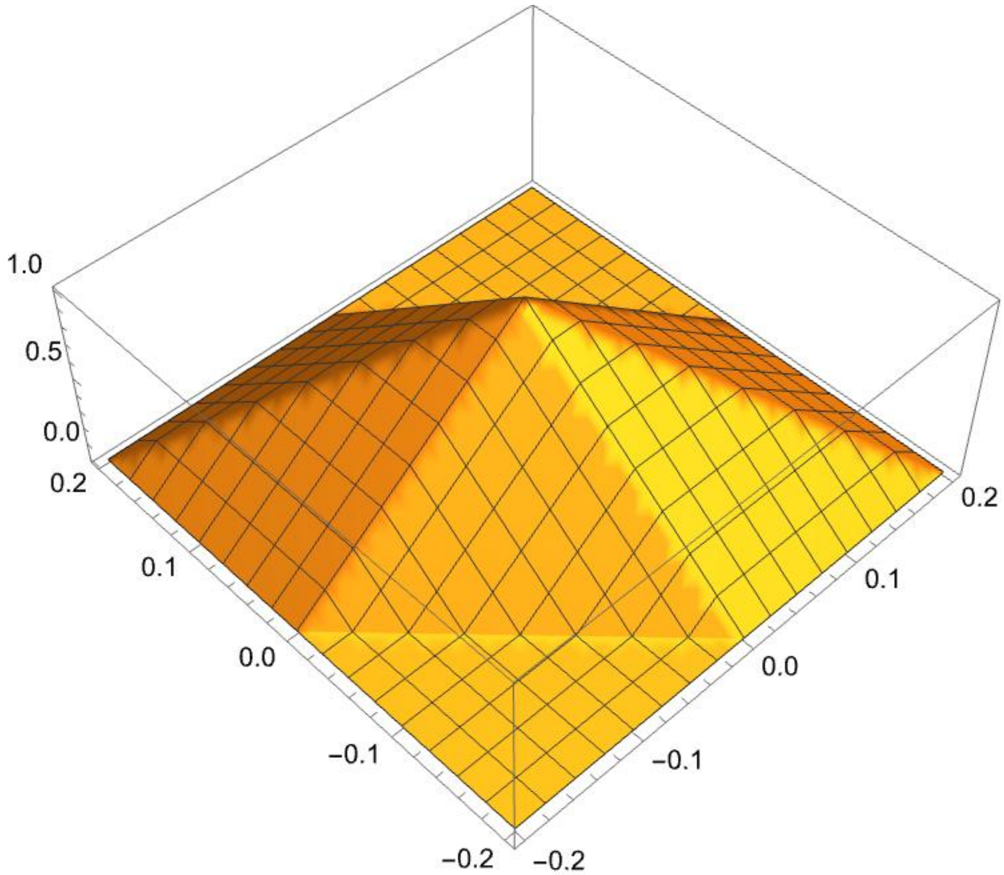

FIGURE 4. The barycentric coordinate $\mu_{(0,0)}$ for $n = 5$

where $D : \mathbb{T}^2 \times \mathbb{T}^2 \to [0, \infty)$ is a metric on the Torus. Let $(i, j) \in V(G_n)$ be given. Since G_n is a Cayley graph, it is vertex transitive. Therefore there is an automorphism $\phi_{(i,j)} : V(G_n) \to V(G_n)$ such that

$$\phi_{(i,j)}(i, j) = (0, 0)$$

We can extend ϕ to the square $[0, 1]^2$ by identifying it with the translation

$$(x, y) \mapsto \left(x - \frac{i}{n}, y - \frac{j}{n}\right)$$

which we denote $\tilde{\phi}_{(i,j)}$. Note that the barycentric coordinate function $\mu_{(i,j)}$ is defined on a hexagonal region identical to the support of $\mu_{(0,0)}$ but centered at $(i/n, j/n)$. Moreover, at any corner $(i, j) + s$, we have

$$\mu_{(i,j)}((i/n, j/n) + s) = \mu_{\phi_{(i,j)}(i,j)}(\tilde{\phi}_{(i,j)}(i/n, j/n) + s) = \mu_{(0,0)}((0, 0) + s)$$

□

Lemma 8. *Let $(i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n$ be given. Then,*

$$\langle \mu_{(0,0)}, \mu_{(i,j)} \rangle = \begin{cases} \frac{1}{2n^2} & \text{if } (i, j) = (0, 0) \\ \frac{1}{12n^2} & \text{if } (i, j) \in S \cup \{(-1, 1), (1, -1)\} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Recall that

$$\langle f, g\rangle = \int_{\mathbb{T}^2} f \wedge *g$$

And

$$*\mu_{(i,j)} = \mu_{(i,j)} dx \wedge dy = \mu_{(i,j)} dx\, dy$$

The result follows by computing the integral

$$\int_0^1 \int_0^1 \mu_{(0,0)}(x, y)\mu_{(0,0)}(x - i/n, y - j/n)\, dx\, dy$$

□

4.2. **Whitney maps of functions on the edges.** Define e_s to be the basis element of $C^1(K, \mathbb{R})$ corresponding to the edge $[(0, 0), s]$. Note that for any $(i, j) \in \mathbb{Z}_n$ the following ordering on the vertices is imposed:

1) $\{v_{(i,j)}\} \preceq \{v_{(i+1,j)}\}$ and $\{v_{(i,j)}\} \preceq \{v_{(i,j+1)}\}$
2) $\{v_{(i-1,j)}\} \preceq \{v_{(i,j)}\}$ and $\{v_{(i,j-1)}\} \preceq \{v_{(i,j)}\}$
3) $\{v_{(i,j)}\} \preceq \{v_{(i+1,j-1)}\}$ and $\{v_{(i-1,j+1)}\} \preceq \{v_{(i,j)}\}$

This is also an example of how S^+ and S^- are constructed in Section 5 Lemma 10.

Remark 3.

$$We_{(0,1)} = \begin{cases} (n^2y)dx + (n - n^2x)dy & \textit{if } (x, y) \in \mathcal{T}_4 \\ (n - n^2y)dx + (n + n^2x)dy & \textit{if } (x, y) \in \mathcal{T}_5 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Note that

$$\mu^{\mathcal{T}_4}_{(0,1)}(x, y) = 1 + n\left(y - \frac{1}{n}\right) \text{ and } \mu^{\mathcal{T}_5}_{(0,1)}(x, y) = 1 + nx + n\left(y - \frac{1}{n}\right)$$

Where $\mu^{\mathcal{T}}$ denotes the restriction of the function μ to the domain $\mathcal{T}$. It follows that

$$d\mu^{\mathcal{T}_4}_{(0,1)}(x, y) = n\, dy \text{ and } d\mu^{\mathcal{T}_5}_{(0,1)}(x, y) = n\, dx + n\, dy$$

Similarly,

$$\mu^{\mathcal{T}_4}_{(0,0)}(x, y) = 1 - ny \text{ and } \mu^{\mathcal{T}_5}_{(0,0)}(x, y) = 1 - nx - ny$$

and

$$d\mu^{\mathcal{T}_4}_{(0,0)}(x, y) = -n\, dx - n\, dy \text{ and } d\mu^{\mathcal{T}_5}_{(0,0)}(x, y) = -n\, dy$$

By definition

$$We_{(0,1)} = \mu_{(0,0)} d\mu_{(0,1)} - \mu_{(0,1)} d\mu_{(0,0)}$$

Simplifying $We^{\mathcal{T}_4}_{(0,1)}$ and $We^{\mathcal{T}}_{(0,1)}$ finishes the proof. □

Remark 4.

$$We_{(1,0)} = \begin{cases} (n + n^2y)dx + (n - n^2x)dy & \textit{if } (x, y) \in \mathcal{T}_3 \\ (n - n^2y)dx + (n^2x)dy & \textit{if } (x, y) \in \mathcal{T}_4 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The proof is nearly identical to the proof of the prior Remark. □

We can now compute $\langle e_{(0,1)}, e_{(1,0)} \rangle$.

$$\begin{aligned}\langle e_{(0,1)}, e_{(1,0)} \rangle &= \int_{\mathcal{T}_4} ((n^2 y)dx + (n - n^2 x)dy) \wedge *((n - n^2 y)dx + (n^2 x)dy) \\ &= \int_{\mathcal{T}_4} ((n^2 y)dx + (n - n^2 x)dy) \wedge ((n - n^2 y)dy - (n^2 x)dx)\end{aligned}$$

which is equal to

$$\int_{\mathcal{T}_4} (n^2 y)(n - n^2 y) dx \wedge dy - \int_{\mathcal{T}_4} (n^2 y)(n^2 x) dx \wedge dx + \int_{\mathcal{T}_4} (n - n^2 x)(n - n^2 y) dy \wedge dy - \int_{\mathcal{T}_4} (n - n^2 x)(n^2 x) dy \wedge dx$$

Since $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -dx \wedge dy$ we have the above expression is equal to

$$\int_{\mathcal{T}_4} n^3 y - n^4 y^2 + n^3 x - n^4 x^2 \, dx \wedge dy = 1/6$$

by evaluating the integral

$$\int_0^{1/n} \int_0^{-y+1/n} n^3 y - n^4 y^2 + n^3 x - n^4 x^2 \, dx dy$$

Remark 5.

$$\langle e_{(0,1)}, e_{(0,1)} \rangle = \frac{2}{3}$$

Proof. The one form $We_{(0,1)}$ is supported on the union of $\mathcal{T}_4$ and $\mathcal{T}_5$ so

$$\int_{\mathbb{T}^2} We_{(0,1)} \wedge *We_{(0,1)} = \int_{\mathcal{T}_4} We_{(0,1)} \wedge *We_{(0,1)} + \int_{\mathcal{T}_5} We_{(0,1)} \wedge *We_{(0,1)}$$

Note that

$$\int_{\mathcal{T}_4} We_{(0,1)} \wedge *We_{(0,1)} = \int_{\mathcal{T}_4} (n^2 y)^2 + (n - n^2 x)^2 \, dx \wedge dy = \frac{1}{3}$$

Moreover,

$$\int_{\mathcal{T}_5} We_{(0,1)} \wedge *We_{(0,1)} = \int_{\mathcal{T}_5} (n - n^2 y)^2 + (n + n^2 x)^2 \, dx \wedge dy = \frac{1}{3}$$

Which proves the result. Once every inner product of the form

$$\langle [(0,0), s_1], [(0,0), s_2] \rangle \quad \text{where} \quad s_1, s_2 \in S$$

has been calculated, it shouldn't be difficult to compute the combinatorial Laplacian on Δ_0. To avoid unnecessarily lengthening the paper, these calculations will be omitted.

□

5. Eigenvectors of the Combinatorial Laplacian

Cayley graphs have many desireable algebraic properties that make them easy to work with. However, they don't encode any geometric information. The motivation for the following definition is to provide a framework for constructing a triangulation with the symmetric properties of a Cayley graph.

Definition 9. *Let $\mathbb{G} = \{e, g_1, \ldots, g_n\}$ be an Abelian group. Let S be an inverse closed subset of $\mathbb{G}$. Let V and E be the vertex and edge set of the Cayley graph of $\mathbb{G}$ with generating set S. Let $K = V \cup E$ be a triangulation of a smooth manifold X. We say that K is a Cayley triangulation if*

1) $\langle v_g, v_{g+s}\rangle_{C^0(K,\mathbb{R})} = \langle v_e, v_s\rangle_{C^0(K,\mathbb{R})}$ *for all* $g \in \mathbb{G}$
2) $\langle (v_x, v_{x+s_1}), (v_y, v_{y+s_2})\rangle_{C^1(K,\mathbb{R})} = \langle (v_e, v_{e+s_1}), (v_{-x+y}, v_{-x+y+s_2})\rangle_{C^1(K,\mathbb{R})}$ *for all* $g \in \mathbb{G}$ *and* $s_1, s_2 \in S$

This definition formalizes the idea that if you define a group action on a Cayley triangulation then the inner product is invariant under that action.

Let X be a manifold. Let $\mathbb{G}$, V, E, S, and K be as defined in definition 9. Let v_g denote element of $C^0(K)$ corresponding to the vertex $\{v_g\}$. Let (x, y) denote the element of $C^1(K)$ corresponding to the edge $\{x, y\}$.

Lemma 9. *Let $x, y \in \mathbb{G}$ be arbitrary. Then,*

$$\langle v_x, v_y\rangle_{C^0(K,\mathbb{R})} = \langle v_e, v_{-x+y}\rangle_{C^0(K,\mathbb{R})}$$

Proof. First note that if x and y are not connected by an edge, then the open star around the point on X corresponding to x and the open star around the point on X corresponding to y are disjoint. Hence, μ_x and μ_y are supported on disjoint sets. Recall that

$$\langle v_x, v_y\rangle = \langle W v_x, W v_y\rangle_{L^2(X)} = \int_X \mu_x \wedge *\mu_y$$

It follows that

$$\langle v_x, v_y\rangle = 0$$

Since x and y are not connected, $-x + y \notin S$. By an identical argument we can conclude that

$$\langle v_e, v_{-x+y}\rangle = 0 = \langle v_x, v_y\rangle$$

On the other hand, if x and y are connected by an edge then $-x + y \in S$. By assumption we have

$$\langle v_e, v_{-x+y}\rangle = \langle v_x, v_{x+-x+y}\rangle = \langle v_x, v_y\rangle$$

$\square$

Lemma 10. *Let $x, y \in \mathbb{G}$ and $s \in S$ be arbitrary. Then,*

$$\langle \delta v_e, (v_{-x+y}, v_{-x+y+s})\rangle = \langle \delta v_x, (v_y, v_{y+s})\rangle$$

Proof. Let $\mathcal{B}_0 = \{\{v_e\}, \{v_{g_1}\}, \ldots, \{v_{g_n}\}\}$. Let $\mathcal{B}_1 = \{\{v_x, v_y\} : -x + y \in S\}$. Recall that

$$M_{\mathcal{B}_0', \mathcal{B}_1'}(\delta) = M_{\mathcal{B}_1, \mathcal{B}_0}(\partial)^t$$

So δ_{v_e} will give us some linear combination of the $C^1(K)$ elements corresponding to the edges which contain v_e as a boundary point. Every edge containing v_e as a boundary is of the form $\{v_e, s\}$ for some $s \in S$. By our choice of orientation, there are elements $S^+ = \{s \in S : g \preceq g + s \text{ for all } g \in \mathbb{G}\}$. Let $S^- = \{s \in S : s \notin S^+\}$. Then

$$\delta v_e = \sum_{s \in S^-} (v_e, v_{e+s}) - \sum_{s \in S^+} (v_e, v_{e+s})$$

Let $x, y \in \mathbb{G}$ and $s_0 \in S$ be given. Note that

$$\langle \delta v_e, (v_{-x+y}, v_{-x+y+s_0})\rangle = \sum_{s\in S^-} \langle (v_e, v_{e+s}), (v_{-x+y}, v_{-x+y+s_0})\rangle - \sum_{s\in S^+} \langle (v_e, v_{e+s}), (v_{-x+y}, v_{-x+y+s_0})\rangle$$

Since K is a Cayley Triangulation, we have that for any $s \in S$

$$\langle (v_e, v_{e+s}), (v_{-x+y}, v_{-x+y+s_0})\rangle = \langle (v_x, v_{x+s}), (v_y, v_{y+s_0})\rangle$$

Therefore,

$$\langle \delta v_e, (v_{-x+y}, v_{-x+y+s_0})\rangle = \langle \delta v_x, (v_y, v_{y+s_0})\rangle$$

Since s_0 was chosen arbitrarily, the result holds. □

Theorem 2. *Let $\mathbb{G} = \{e, g_1, \ldots, g_n\}$ be a finite Abelian group. Let K be a triangulation of a manifold X whose edges and vertices are given by the Cayley graph of $\mathbb{G}$ with a generating set S. If K is a Cayley triangulation of X then every character of $\mathbb{G}$ is an eigenvector of the Combinatorial Laplacian on $C^0(K)$.*

Proof. First note that the matrix of δ^* satisfies

$$M(\delta^*) = C^{-1}R$$

where

$$C_{i,j} = \langle v_{g_i}, v_{g_j}\rangle \text{ and } R_{i,j} = \langle \delta v_{g_i}, E_j\rangle$$

First define $f : \mathbb{G} \to \mathbb{R}$ by

$$f(g) = \langle v_e, v_g\rangle$$

Note that,

$$f(-g_i + g_j) = \langle v_e, v_{-g_i+g_j}\rangle$$

By assumption

$$\langle v_e, v_{-g_i+g_j}\rangle = \langle v_{g_i}, v_{g_j}\rangle \tag{12}$$

Thus,

$$f(-g_i + g_j) = \langle v_{g_i}, v_{g_j}\rangle$$

It follows that the characters of $\mathbb{G}$ are eigenvectors of C.

We must now prove that $RM(\partial)^t$ is a Cayley matrix. That is, we need to show that there exists an inverse invariant function $f : \mathbb{G} \to \mathbb{R}$ such that

$$f(-g_i + g_j) = RM(\partial)^t_{i,j}$$

Let $S^+ = \{s : v_g \preceq v_{g+s}\}$. (See the beginning of Section 4) Let $S^- = S\backslash S^+$. Define $f : \mathbb{G} \to \mathbb{R}$ by

$$f(g) = \sum_{s\in S^+} \langle \delta v_e, (v_g, v_{g+s})\rangle - \sum_{s\in S^-} \langle \delta v_e, (v_g, v_{g+s})\rangle$$

Note that

$$\begin{aligned} f(-g_i + g_j) &= \sum_{s\in S^+} \langle \delta v_e, (v_{-g_i+g_j}, v_{-g_i+g_j+s})\rangle - \sum_{s\in S^-} \langle \delta v_e, (v_{-g_i+g_j}, v_{-g_i+g_j+s})\rangle \\ &= \sum_{s\in S^+} \langle \delta v_{g_i}, (v_{g_j}, v_{g_j+s})\rangle - \sum_{s\in S^-} \langle \delta v_{g_i}, (v_{g_j}, v_{g_j+s})\rangle \\ &= \sum_{\{x,y\}\in E} \langle \delta v_{g_i}, (x, y)\rangle \alpha_j^{(x,y)} \end{aligned}$$

where for any edge $\{x, y\}$ we have

$$\partial\{x, y\} = \alpha_1^{(x,y)}\{v_{g_1}\} + \ldots + \alpha_n^{(x,y)}\{v_{g_n}\}$$

Now, enumerating the edges $E_1, \ldots, E_m$ we have

$$f(-g_i + g_j) = \sum_{k=1}^{m} \langle \delta v_{g_i}, E_k \rangle \alpha_j^{E_k} = RM(\partial)^t_{i,j}$$

Hence, $RM(\partial)^t$ is a Cayley matrix of $\mathbb{G}$. By Lemma 1, it follows that the characters of $\mathbb{G}$ are eigenvectors of $RM(\partial)^t$.

Let $\chi \in \widehat{\mathbb{G}}$ be given. Then,

$$\begin{aligned} M(\Delta_c)\chi &= C^{-1} RM(\partial)\chi \\ &= C^{-1}\lambda_{R\partial}\chi \\ &= \lambda_C^{-1}\lambda_{R\partial}\chi \end{aligned}$$

where $C\chi = \lambda_C \chi$. Since χ was chosen arbitrarily, it follows that each character is an eigenvector of $M(\Delta_c)$ □

Note that in this proof, we never actually use the inner product from the De Rham chain complex. So we can define d^* using any inner product and the resulting combinatorial Laplacian will have the same eigenvectors as the graph Laplacian. Equivalently, we can say that the eigenspaces of the combinatorial Laplacian are invariant under choice of inner product on $C^q(K)$.

Corollary 2. *Let $\mathbb{G} = \{e, g_1, \ldots, g_n\}$ be a finite Abelian group. Let K be a Cayley triangulation of a manifold X. Then,*

$$\Delta_c \Delta_G = \Delta_G \Delta_c$$

Proof. Since Δ_c and Δ_G have the same eigenspaces, they commute. □

A natural question to ask is for what types of manifolds can we apply a Cayley triangulation. It is likely that only Tori admit Cayley triangulations. If $\mathbb{G}$ is an arbitrary finite abelian group then $\mathbb{G}$ is isomorphic to a product of cyclic groups. So every Cayley triangulation would have the structure of a Torus.

It is also important to ask: what about this goes wrong if we do not have a Cayley triangulation? In particular, what goes wrong if we embed a Cayley graph such that it is not a Cayley triangulation? Cayley triangulations are essentially Cayley graphs whose vertices are placed "uniformly" on the manifold. The inner product on the co-chain complex comes from the inner product on the De Rham complex, which is given by integration over the manifold. However, the barycentric coordinates are supported only on certain regions of the manifold. So verifying that

$$\langle v_e, v_s \rangle = \langle v_g, v_{g+s} \rangle$$

for all $g \in \mathbb{G}$ and $s \in S$ tells us that we can map the support of Wv_e into the support of Wv_g with a translation.

6. Future Goals

Theorem 2 shows that there is a connection between the fundamental group of a manifold and the eigenvectors of the combinatorial Laplacian constructed using a triangulation from the Cayley graph of that fundamental group–provided that the fundamental group is Abelian. We could also view this connection as a connection to a discrete group of isometries on the surface - self-maps on the surface which preserve the length. This begs the question, what if the fundamental group is non-Abelian? Or, what happens if we wish to study a space whose group of isometries is not abelian? The key result that allowed us to make the connection between the group structure of tori and the combinatorial Laplacian is Lemma 1. This lemma can be generalized to the case when the group is non-Abelian, however. It is possible that the eigenvalues of the combinatorial Laplacian on manifolds with a non-Abelian fundamental groups can be computed using this more general result. However, the fundamental group of a manifold only encodes topological information about the manifold, so it might not be possible to embed a Cayley graph of the fundamental group in a way which respects the geometry of the manifold.

There are also other lingering questions. For instance, it appears that the combinatorial Laplacian commutes with the action of a discrete group on the embedded graph. This is to be expected though, as trace formulas in harmonic analysis crop up when operators commute with group actions. This begs the question: if we have a more general group action on a graph that is a discrete group of a group action on the manifold, can once construct discrete operators that commute with this action? What do these operators look like in the limit?

References

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