PATTERNS IN THE CAHN-HILLIARD EQUATION WITH LONG-RANGE INTERACTIONS

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Abstract. The Cahn-Hilliard equation is a partial differential equation that governs the behavior of a binary fluid system. In this work, we use a version of the Cahn-Hilliard equation that contains an additional term to account for the long-range interaction of the fluid molecules. We analyze the dynamic transitions and pattern formation of the model as we vary a system control parameter λ. One of the main goals of this work is to deduce necessary and sufficient conditions (on λ and fixed parameters) for the equilibria to form hexagonally packed cylinder (HPC) patterns.

1. Introduction

In this work, we consider the Cahn-Hilliard equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} = -\Delta^2 u - \lambda\Delta u + \Delta(\gamma_2 u^2 + \gamma_3 u^3) - \sigma u \\
&\int_\Omega u(x,t,\lambda)dx = 0 \\
&\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial\Omega \\
&u(x,0,\lambda) = \psi.
\end{aligned} \tag{1}$$

on a rectangular domain $\Omega := [0, L_1] \times [0, L_2] \times [0, L_3]$. The Cahn-Hilliard equation describes the behavior of a binary fluid system, where u is a function that denotes the deviation from the average concentration of one component and λ, γ_2, γ_3, and σ are parameters that depend on physical properties of the system, such as temperature. This problem is analyzed in-depth in [1]. We analyze the Cahn-Hilliard equation using dynamic transition theory (discussed in [1], [2], [3], [4]) which studies how the equilibrium behavior of a differential equation changes as we vary a parameter λ. We assume that the following principle of exchange of stability (PES) holds:

Principle 1.1. Suppose

$$\begin{aligned}
&\frac{du}{dt} = L_\lambda u + G(u,\lambda), \\
&u(0) = u_0,
\end{aligned}$$

where L_λ is a linear operator and G consists of higher-order terms in u. Let $\{\beta_i(\lambda) \in \mathbb{C} : i \in \mathbb{N}\}$ be the set of eigenvalues of L_λ counting multiplicities. Then

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$$
\begin{aligned}
&\mathrm{Re}(\beta_j(\lambda)) \begin{cases} < 0, & \text{if } \lambda < \lambda_c, \\ = 0, & \text{if } \lambda = \lambda_c, \\ > 0, & \text{if } \lambda > \lambda_c, \end{cases} && \forall 1 \leq j \leq m \\
&\mathrm{Re}(\beta_j(\lambda)) < 0, && \forall j \geq m+1
\end{aligned}
$$

We now state a theorem that appears in [2] as Theorem 2.1.3:

Theorem 1.2. *Let H be the codomain of u in (1), and assume that H is a Hilbert space. Suppose that the PES holds. Then the problem 1 always undergoes a dynamic transition from $(u, \lambda) = (0, \lambda_c)$, and there is a neighborhood $U \subset H$ of $u = 0$ such that the transition in U is one of the following three types:*

(i) Continuous (Type-I) transition: There exists an open and dense set $\tilde{U}_\lambda \subset U$ such that for any $\phi \in \tilde{U}_\lambda$, the solution $u_\lambda(t, \phi)$ of (1) with initial datum $u_\lambda(0, \lambda)$ satisfies

$$\lim_{\lambda \to \lambda_c} \limsup_{t \to \infty} ||u_\lambda(t, \phi)|| = 0.$$

(ii) Jump (Type-II) transition: For any $\lambda_c < \lambda < \lambda + \epsilon$ with some $\epsilon > 0$, there is an open and dense set $\tilde{U}_\lambda \subset U$ such that for any $\phi \in \tilde{U}_\lambda$,

$$\limsup_{t \to \infty} ||u_\lambda(t, \phi)|| \geq \delta > 0,$$

where $\delta > 0$ is independent of λ.

(iii) Mixed (Type-III) transition: For any $\lambda_c < \lambda < \lambda + \epsilon$ with some $\epsilon > 0$, U can be decomposed into two open (not necessarily connected) sets U_1^λ and U_2^λ:

$$\overline{U} = \overline{U_1^\lambda} \cup \overline{U_2^\lambda}, \quad U_1^\lambda \cap U_2^\lambda = \varnothing$$

such that

$$
\begin{aligned}
&\lim_{\lambda \to \lambda_c} \limsup_{t \to \infty} ||u_\lambda(t, \phi)|| = 0 \quad \forall \phi \in U_1^\lambda \\
&\limsup_{t \to \infty} ||u_\lambda(t, \phi)|| \geq \delta > 0 \quad \forall \phi \in U_2^\lambda.
\end{aligned}
$$

One of our main goals in this work is to determine conditions on γ_2, γ_3, σ, L_1, L_2, and L_3 under which each of the above types of dynamic transitions occur. Also, we seek to determine the conditions under which some of the equilibria generate hexagonally packed cylinder (HPC) patterns.

2. Main Results

Definition 2.1. We define the set $\mathcal{P}$ as follows:

$$\mathcal{P} := \left\{ \left(\frac{k_1 \pi}{L_1}, \frac{k_2 \pi}{L_2}, \frac{k_3 \pi}{L_3} \right) \quad \middle| \quad k_i \in \mathbb{N}_0, \quad 1 \leq i \leq 3, \quad \sum_{i=1}^{3} k_i^2 \neq 0 \right\}.$$

For a vector $K \in \mathcal{P}$, we let

$$e_K(x_1, x_2, x_3) := \cos\left(\frac{k_1 x_1}{L_1}\right) \cos\left(\frac{k_2 x_2}{L_2}\right) \cos\left(\frac{k_3 x_3}{L_3}\right).$$

Observe that e_K is an eigenfunction of the Laplacian with eigenvalue $-|K|^2$.

Proposition 2.2. *For distinct $K_1, K_2 \in \mathcal{P}$, we have*

$$\int_\Omega e_{K_1}(x)e_{K_2}(x)dx = 0,$$

where $dx = dx_1dx_2dx_3$.

Now note that in (1), $L_\lambda = -\Delta^2 - \lambda\Delta - \sigma I$. Therefore, e_K is an eigenfunction of L_λ with eigenvalue

$$\beta_K(\lambda) = -|K|^4 + \lambda|K|^2 - \sigma = |K|^2\left(\lambda - \frac{|K|^4+\sigma}{|K|^2}\right).$$

Therefore, PES implies that

$$\lambda_c = \min_{K\in\mathcal{P}} \frac{|K|^4+\sigma}{|K|^2}.$$

Definition 2.3. We define the set $\mathcal{S}$ as follows:

$$\mathcal{S} = \left\{K \in \mathcal{P} : \frac{|K|^4+\sigma}{|K|^2} = \lambda_c\right\}.$$

In [1], the following theorem is presented:

Theorem 2.4. *Suppose $L_1 = 2\pi L$, $L_2 = \frac{2}{\sqrt{3}}\pi L$, and $L_3 = \theta\pi L$ for some $L, \theta > 0$ depending on σ. Assume that $K_1^c = \left(\frac{n}{L}, 0, 0\right)$ and $K_2^c = \left(\frac{n}{2L}, \frac{\sqrt{3}n}{2L}, 0\right)$ are such that $\mathcal{S} = \{K_1^c, K_2^c\}$. Let*

$$\mathcal{B} := \gamma_3 - \frac{8|K_1^c|^2}{36|K_1^c|^4 - 9\sigma}\gamma_2^2.$$

(i) *If $\gamma_2 = 0$, then the phase transition of (1) at λ_c is Type-I. The problem bifurcates on the side $\lambda > \lambda_c$ to an attractor Σ_λ, which is homeomorphic to the one-dimensional unit sphere S^1. Σ_λ contains eight non-degenerate steady states, with four saddle points v_1, v_2, v_3, and v_4 and four minimal attractors u_1, u_2, u_3, and u_4. Moreover, the following approximation formulas hold:*

$$u_{1,3} = \pm\sqrt{\frac{4\beta_1(\lambda)}{3|K_1^c|^2\gamma_3}}\cos\left(\frac{nx_1}{L}\right) + o(|\beta_1(\lambda)|^{1/2})$$

$$u_{2,4} = \pm\sqrt{\frac{16\beta_1(\lambda)}{9|K_1^c|^2\gamma_3}}\cos\left(\frac{nx_1}{2L}\right)\cos\left(\frac{\sqrt{3}nx_2}{2L}\right) + o(|\beta_1(\lambda)|^{1/2})$$

$$v_{1,2,3,4} = \pm\sqrt{\frac{4\beta_1(\lambda)}{15|K_1^c|^2\gamma_3}}\cos\left(\frac{nx_1}{L}\right) \pm 2\sqrt{\frac{4\beta_1(\lambda)}{15|K_1^c|^2\gamma_3}}\cos\left(\frac{nx_1}{2L}\right)\cos\left(\frac{\sqrt{3}nx_2}{2L}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

where $\beta_1(\lambda) := \beta_{K_1^c}(\lambda)$.

(ii) If $\gamma_2 \neq 0$ and $\mathcal{B} < 0$, then (1) bifurcates on both sides of λ_c and the transition is Type-II. Moreover, there are four steady states bifurcated out on the side $\lambda < \lambda_c$, including three saddle points and one unstable node. On the side $\lambda > \lambda_c$, the problem bifurcates to two steady states, which are saddles.

(iii) If $\gamma_2 \neq 0$ and $\mathcal{B} > 0$, then the transition is Type-III. Again, there are bifurcations on both sides of λ_c. On the side $\lambda < \lambda_c$, there are two saddles bifurcating out from the origin. On the side $\lambda > \lambda_c$, the problem bifurcates to four steady states:

$$w_1 = \frac{\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{nx_1}{L}\right) + \frac{2\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{nx_1}{2L}\right)\cos\left(\frac{\sqrt{3}nx_2}{2L}\right) + o(|\beta_1(\lambda)|),$$

$$w_2 = \frac{\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{nx_1}{L}\right) - \frac{2\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{nx_1}{2L}\right)\cos\left(\frac{\sqrt{3}nx_2}{2L}\right) + o(|\beta_1(\lambda)|),$$

$$w_3 = \sqrt{-\frac{\beta_1(\lambda)}{b(\lambda)}}\cos\left(\frac{nx_1}{L}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

$$w_4 = -\sqrt{-\frac{\beta_1(\lambda)}{b(\lambda)}}\cos\left(\frac{nx_1}{L}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

where $b(\lambda) = \frac{2|K_1^c|^4\gamma_2^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma} - \frac{3|K_1^c|^2}{4}\gamma_3$. Among the four steady states, there are three saddles and one stable node, where the node is w_3 if $\gamma_2 > 0$ and w_4 if $\gamma_2 < 0$.

We extend this result as follows:

Theorem 2.5. *Suppose that the side lengths of Ω satisfy $L_1 = \frac{m_1\sqrt{3}L_2}{m_2}$, where $m_1, m_2 \in \mathbb{N}$. Suppose that $K_1^c := \left(\frac{2m_1\pi}{L_1}, 0, 0\right)$ and $K_2^c := \left(\frac{m_1\pi}{L_1}, \frac{m_2\pi}{L_2}, 0\right)$ are such that $\mathcal{S} = \{K_1^c, K_2^c\}$. Then we can derive results analogous to the previous theorem:*

(i) If $\gamma_2 = 0$, then the phase transition of (1) at λ_c is Type-I. The problem bifurcates on the side $\lambda > \lambda_c$ to an attractor Σ_λ, which is homeomorphic to the one-dimensional unit sphere S^1. Σ_λ contains eight non-degenerate steady states, with four saddle points v_1, v_2, v_3, and v_4 and four minimal attractors u_1, u_2, u_3, and u_4. Moreover, the following approximation formulas hold:

$$u_{1,3} = \pm\sqrt{\frac{4\beta_1(\lambda)}{3|K_1^c|^2\gamma_3}}\cos\left(\frac{2m_1\pi x_1}{L_1}\right) + o(|\beta_1(\lambda)|^{1/2})$$

$$u_{2,4} = \pm\sqrt{\frac{16\beta_1(\lambda)}{9|K_1^c|^2\gamma_3}}\cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) + o(|\beta_1(\lambda)|^{1/2})$$

$$v_{1,2,3,4} = \pm\sqrt{\frac{4\beta_1(\lambda)}{15|K_1^c|^2\gamma_3}}\cos\left(\frac{2m_1\pi x_1}{L_1}\right) \pm 2\sqrt{\frac{4\beta_1(\lambda)}{15|K_1^c|^2\gamma_3}}\cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

where $\beta_1(\lambda) := \beta_{K_1^c}(\lambda)$.

(ii) If $\gamma_2 \neq 0$ and $\mathcal{B} < 0$, then (1) bifurcates on both sides of λ_c and the transition is Type-II. Moreover, there are four steady states bifurcated out on the side $\lambda < \lambda_c$, including three saddle points and one unstable node. On the side $\lambda > \lambda_c$, the problem bifurcates to two steady states, which are saddles.

(iii) If $\gamma_2 \neq 0$ and $\mathcal{B} > 0$, then the transition is Type-III. Again, there are bifurcations on both sides of λ_c. On the side $\lambda < \lambda_c$, there are two saddles bifurcating out from the origin. On the side $\lambda > \lambda_c$, the problem bifurcates to four steady states:

$$w_1 = \frac{\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{m_1\pi x_1}{L_1}\right) + \frac{2\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) + o(|\beta_1(\lambda)|),$$

$$w_2 = \frac{\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{m_1\pi x_1}{L_1}\right) - \frac{2\beta_1(\lambda)}{|K_1^c|^2\gamma_2}\cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) + o(|\beta_1(\lambda)|),$$

$$w_3 = \sqrt{-\frac{\beta_1(\lambda)}{b(\lambda)}}\cos\left(\frac{2m_1\pi x_1}{L_1}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

$$w_4 = -\sqrt{-\frac{\beta_1(\lambda)}{b(\lambda)}}\cos\left(\frac{2m_1\pi x_1}{L_1}\right) + o(|\beta_1(\lambda)|^{1/2}),$$

where $b(\lambda) = \frac{2|K_1^c|^4\gamma_2^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma} - \frac{3|K_1^c|^2}{4}\gamma_3$. Among the four steady states, there are three saddles and one stable node, where the node is w_3 if $\gamma_2 > 0$ and w_4 if $\gamma_2 < 0$.

Proof. We use e_1 and e_2 as shorthand notation for $e_{K_1^c}$ and $e_{K_2^c}$, respectively. Thus,

$$e_1 = \cos\left(\frac{2m_1\pi x_1}{L_1}\right), \quad e_2 = \cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right).$$

The critical eigenspace is $H_c = \text{span}\{e_1, e_2\}$, and the stable eigenspace is the orthogonal complement of H_c in H. Since

$$H = \text{span}\{e_K : K \in \mathcal{P}\},$$

we have

$$H_s = \text{span}\{e_K : K \in \mathcal{P} \setminus \{K_1^c, K_2^c\}\}$$

We can write our solution to (1) on the center manifold as

$$u(x,t,\lambda) = v(x,t,\lambda) + \Psi(v(x,t,\lambda),\lambda),$$

where $v \in H_c$, i.e.,

$$v(x,t,\lambda) = y_1(t,\lambda)e_1(x) + y_2(t,\lambda)e_2(x),$$

and $\Psi : H_c \to H_s$ is the center manifold function. Since Ψ maps to H_s, we have that

$$\Phi(y_1e_1 + y_2e_2, \lambda) = \text{proj}_{H_s}\Phi(y_1e_1 + y_2e_2, \lambda) \tag{2}$$

$$= \sum_{K \in \mathcal{P}\setminus\{K_1^c, K_2^c\}} \frac{\langle \Phi(y_1e_1 + y_2e_2, \lambda), e_K\rangle}{\langle e_K, e_K\rangle} e_K. \tag{3}$$

We also have

$$\Phi(y_1e_1 + y_2e_2, \lambda) = (-L_\lambda^s)^{-1}P_sG_2(y_1e_1 + y_2e_2) + o(2),$$

where $G_2(v) = \gamma_2 \Delta v^2$ and

$$o(k) := O(|\beta_{K_1}(\lambda)||y|^k) + o(|y|^k).$$

Now note that for each $K \in \mathcal{P} \setminus \{K_1, K_2\}$,

$$\begin{aligned}\langle \Phi(y_1e_1 + y_2e_2, \lambda), e_K\rangle &= \langle (-L_\lambda^s)^{-1} P_s G_2(y_1e_1 + y_2e_2), e_K\rangle + o(2)\\ &= \langle P_s G_2(y_1e_1 + y_2e_2), (-L_\lambda^s)^{-1} e_K\rangle + o(2)\\ &= -\frac{1}{\beta_K(\lambda)}\langle G_2(y_1e_1 + y_2e_2), e_K\rangle + o(2).\end{aligned}$$

Substituting into (2), we obtain

$$\Phi(y_1e_1 + y_2e_2, \lambda) = \sum_{K\in\mathcal{P}\setminus\{K_1^c,K_2^c\}} -\frac{\langle G_2(y_1e_1 + y_2e_2), e_K\rangle}{\beta_K(\lambda)\langle e_K, e_K\rangle} e_K + o(2).$$

We will now work to simplify the expression $\langle G_2(y_1e_1 + y_2e_2), e_K\rangle$. We have

$$\begin{aligned}\langle G_2(y_1e_1 + y_2e_2), e_K\rangle &= \gamma_2 \left\langle \Delta\left((y_1e_1 + y_2e_2)^2\right), e_K\right\rangle\\ &= \gamma_2 \left\langle (y_1e_1 + y_2e_2)^2, \Delta e_K\right\rangle\\ &= -\gamma_2|K|^2 \left\langle (y_1e_1 + y_2e_2)^2, e_K\right\rangle\\ &= -\gamma_2|K|^2 \left(y_1^2 \left\langle e_1^2, e_K\right\rangle + 2y_1y_2 \left\langle e_1e_2, e_K\right\rangle + y_2^2 \left\langle e_2^2, e_K\right\rangle\right).\end{aligned}$$

Since

$$e_1^2 = \cos^2\left(\frac{2m_1\pi x_1}{L_1}\right) = \frac{1 + \cos\left(\frac{4m_1\pi x_1}{L_1}\right)}{2},$$

we have (for $K \in \mathcal{P} \setminus \{K_1^c, K_2^c\}$)

$$\left\langle e_1^2, e_K\right\rangle = \frac{1}{2}\int_\Omega \left(1 + \cos\left(\frac{4m_1\pi x_1}{L_1}\right)\right) e_K dx = \begin{cases}\frac{L_1L_2L_3}{4}, & K = K_1 := \left(\frac{4m_1\pi}{L_1}, 0, 0\right)\\ 0, & \text{otherwise.}\end{cases}$$

Similarly,

$$\begin{aligned}\langle e_1e_2, e_K\rangle &= \int_\Omega \cos\left(\frac{2m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) e_K dx\\ &= \frac{1}{2}\int_\Omega \cos\left(\frac{m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) e_K dx\\ &+ \frac{1}{2}\int_\Omega \cos\left(\frac{3m_1n\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right) e_K dx\\ &= \begin{cases}\frac{L_1L_2L_3}{8}, & K = K_2 := \left(\frac{3m_1\pi}{L_1}, \frac{m_2\pi}{L_2}, 0\right)\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Finally,

$$\begin{aligned}\left\langle e_2^2, e_K\right\rangle &= \int_\Omega \cos^2\left(\frac{m_1\pi x_1}{L_1}\right)\cos^2\left(\frac{m_2\pi x_2}{L_2}\right) e_K dx\\ &= \frac{1}{4}\int_\Omega \left(1 + \cos\left(\frac{2m_1\pi x_1}{L_1}\right)\right)\left(1 + \cos\left(\frac{2m_2\pi x_2}{L_2}\right)\right) e_K dx\end{aligned}$$

$$= \begin{cases} \frac{L_1L_2L_3}{8}, & K = K_3 := \left(0, \frac{2m_2\pi}{L_2}, 0\right) \\ \frac{L_1L_2L_3}{16}, & K = K_4 := \left(\frac{2m_1\pi}{L_1}, \frac{2m_2\pi}{L_2}, 0\right) \\ 0, & \text{otherwise.} \end{cases}$$

Since $L_1 = \frac{m_1\sqrt{3}L_2}{m_2}$, we calculate

$$\begin{aligned} |K_1^c|^2 = |K_2^c|^2 &= \frac{4m_2^2\pi^2}{3L_2^2} \\ |K_1|^2 = |K_4|^2 &= \frac{16m_2^2\pi^2}{3L_2^2} = 4|K_1^c|^2 \\ |K_2|^2 = |K_3|^2 &= \frac{4m_2^2\pi^2}{L_2^2} = 3|K_1^c|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \Phi(y_1e_1 + y_2e_2, \lambda) &= -\sum_{i=1}^{4} \frac{\langle G_2(y_1e_1 + y_2e_2, \lambda), e_{K_i}\rangle}{\beta_{K_i}(\lambda)\langle e_{K_i}, e_{K_i}\rangle} e_{K_i} \\ &= -\sum_{i=1}^{4} \frac{\gamma_2|K_i|^2 \left(y_1^2 \langle e_1^2, e_{K_i}\rangle + 2y_1y_2 \langle e_1e_2, e_{K_i}\rangle + y_2^2 \langle e_2^2, e_{K_i}\rangle\right)}{\left(|K_i|^4 - \lambda|K_i|^2 + \sigma\right) \langle e_{K_i}, e_{K_i}\rangle} e_{K_i} \\ &= -\frac{4\gamma_2|K_1^c|^2 y_1^2 \left(\frac{L_1L_2L_3}{4}\right)}{\left(16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma\right)\left(\frac{L_1L_2L_3}{2}\right)} \cos\left(\frac{4m_1\pi x_1}{L_1}\right) \\ &\quad - \frac{3\gamma_2|K_1^c|^2 2y_1y_2 \left(\frac{L_1L_2L_3}{8}\right)}{\left(9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma\right)\left(\frac{L_1L_2L_3}{4}\right)} \cos\left(\frac{3m_1\pi x_1}{L_1}\right) \cos\left(\frac{m_2\pi x_2}{L_2}\right) \\ &\quad - \frac{3\gamma_2|K_1^c|^2 y_2^2 \left(\frac{L_1L_2L_3}{8}\right)}{\left(9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma\right)\left(\frac{L_1L_2L_3}{2}\right)} \cos\left(\frac{2m_2\pi x_2}{L_2}\right) \\ &\quad - \frac{4\gamma_2|K_1^c|^2 y_2^2 \left(\frac{L_1L_2L_3}{16}\right)}{\left(16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma\right)\left(\frac{L_1L_2L_3}{4}\right)} \cos\left(\frac{2m_1\pi x_1}{L_1}\right) \cos\left(\frac{2m_2\pi x_2}{L_2}\right) \\ &= -\frac{2\gamma_2|K_1^c|^2 y_1^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma} \cos\left(\frac{4m_1\pi x_1}{L_1}\right) \\ &\quad - \frac{3\gamma_2|K_1^c|^2 y_1y_2}{9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma} \cos\left(\frac{3m_1\pi x_1}{L_1}\right) \cos\left(\frac{m_2\pi x_2}{L_2}\right) \\ &\quad - \frac{3\gamma_2|K_1^c|^2 y_2^2}{4\left(9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma\right)} \cos\left(\frac{2m_2\pi x_2}{L_2}\right) \\ &\quad - \frac{\gamma_2|K_1^c|^2 y_2^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma} \cos\left(\frac{2m_1\pi x_1}{L_1}\right) \cos\left(\frac{2m_2\pi x_2}{L_2}\right) \end{aligned}$$

We now let

$$\begin{aligned} A_1(\lambda) &:= \frac{2\gamma_2|K_1^c|^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma} \\ A_2(\lambda) &:= \frac{3\gamma_2|K_1^c|^2}{9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma} \end{aligned}$$

$$A_3(\lambda) := \frac{3\gamma_2|K_1^c|^2}{4\left(9|K_1^c|^4 - 3\lambda|K_1^c|^2 + \sigma\right)}$$
$$A_4(\lambda) := \frac{\gamma_2|K_1^c|^2}{16|K_1^c|^4 - 4\lambda|K_1^c|^2 + \sigma}$$

so that

$$\begin{aligned}\Phi(y_1e_1 + y_2e_2, \lambda) = &-A_1(\lambda)y_1^2 \cos\left(\frac{4m_1\pi x_1}{L_1}\right)\\ &- A_2(\lambda)y_1y_2 \cos\left(\frac{3m_1\pi x_1}{L_1}\right)\cos\left(\frac{m_2\pi x_2}{L_2}\right)\\ &- A_3(\lambda)y_2^2 \cos\left(\frac{2m_2\pi x_2}{L_2}\right)\\ &- A_4(\lambda)y_2^2 \cos\left(\frac{2m_1\pi x_1}{L_1}\right)\cos\left(\frac{2m_2\pi x_2}{L_2}\right)\end{aligned}$$

We now take the Cahn-Hilliard Equation (1) and take the inner product of both sides with e_1. Since $u = y_1e_1 + y_2e_2 + \Phi(y_1e_1 + y_2e_2, \lambda)$, we have

$$\frac{dy_1}{dt} = \beta_{K_1^c}(\lambda)y_1 + \frac{1}{\langle e_1, e_1\rangle}\int_\Omega \Delta\left(\gamma_2u^2 + \gamma_3u^3\right)e_1dx \tag{4}$$
$$= \beta_{K_1^c}(\lambda)y_1 + \frac{2}{L_1L_2L_3}\int_\Omega \left(\gamma_2u^2 + \gamma_3u^3\right)\Delta e_1dx \tag{5}$$
$$= \beta_{K_1^c}(\lambda)y_1 - \frac{2|K_1^c|^2}{L_1L_2L_3}\int_\Omega \left(\gamma_2u^2 + \gamma_3u^3\right)e_1dx \tag{6}$$

We evaluate this integral using Maple (calculations included in the appendix). We obtain, up to third-order terms,

$$\begin{aligned}\frac{dy_1}{dt} = \beta_{K_1^c}(\lambda)y_1 - 2|K_1^c|^2\Bigg[&\left(\frac{\gamma_2}{8}\right)y_2^2\\ &+\left(\frac{3\gamma_3}{8} - \frac{A_1(\lambda)\gamma_2}{2}\right)y_1^3\\ &+\left(\frac{3\gamma_3}{8} - \frac{A_2(\lambda)\gamma_2}{4}\right)y_1y_2^2\Bigg] + o(3).\end{aligned}$$

We define

$$B(\lambda) := -2|K_1^c|^2\left(\frac{3\gamma_3}{8} - \frac{A_1(\lambda)\gamma_2}{2}\right)$$
$$C(\lambda) := -2|K_1^c|^2\left(\frac{3\gamma_3}{8} - \frac{A_2(\lambda)\gamma_2}{4}\right)$$

so that

$$\frac{dy_1}{dt} = \beta_{K_1^c}(\lambda)y_1 - \frac{\gamma_2|K_1^c|^2}{4}y_2^2 + B(\lambda)y_1^3 + C(\lambda)y_1y_2^2 + o(3)$$

In a very similar manner, we obtain

$$\frac{dy_2}{dt} = \beta_{K_1^c}(\lambda)y_2 - \gamma_2|K_1^c|^2y_1y_2 + D(\lambda)y_2^3 + E(\lambda)y_1^2y_2 + o(3),$$

where

$$D(\lambda) := -4|K_1^c|^2\left(\frac{9\gamma_3}{64} - \frac{(2A_3(\lambda)+A_4(\lambda))\gamma_2}{8}\right)$$
$$E(\lambda) := -4|K_1^c|^2\left(\frac{3\gamma_3}{8} - \frac{A_2(\lambda)\gamma_2}{4}\right).$$

Thus, the system of interest is (up to third-order terms)

$$\begin{aligned}\frac{dy_1}{dt} &= \beta_{K_1^c}(\lambda)y_1 - \frac{\gamma_2|K_1^c|^2}{4}y_2^2 + B(\lambda)y_1^3 + C(\lambda)y_1y_2^2\\ \frac{dy_2}{dt} &= \beta_{K_1^c}(\lambda)y_2 - \gamma_2|K_1^c|^2y_1y_2 + D(\lambda)y_2^3 + E(\lambda)y_1^2y_2.\end{aligned} \tag{7}$$

Note that the $\gamma_2 = 0$ case simplifies (7) to a case already discussed in the paper with a different value of $|K_1^c|^2$. However, the same method still applies. Also, because the quadratic terms are also identical to the case discussed in [1], the analysis provided for this case also applies here. In particular, we have that when $y_2 = 2y_1$,

$$\frac{\frac{dy_1}{dt}}{\frac{dy_2}{dt}} = \frac{\beta_{K_1^c}(\lambda)y_1 - \gamma_2|K_1^c|^2y_1^2 + B(\lambda)y_1^3 + 4C(\lambda)y_1^3}{2\beta_{K_1^c}(\lambda)y_1 - 2\gamma_2|K_1^c|^2y_1^2 + 8D(\lambda)y_1^3 + 2E(\lambda)y_1^3}. \tag{8}$$

As is the case in [1], we have that

$$\frac{B(\lambda)+4C(\lambda)}{8D(\lambda)+2E(\lambda)} = \frac{1}{2}.$$

Therefore, (8) simplifies to $\frac{1}{2}$ as well, so the analysis for the $\gamma_2 \neq 0$ case in the paper is also valid here. □

We now provide a proposition that provides a more detailed classification of when $\mathcal{S} = \{K_1^c, K_2^c\}$, where K_1^c and K_2^c are as defined in the previous theorem.

Proposition 2.6. *Suppose that* $L_1 = \frac{m_1\sqrt{3}L_2}{m_2}$, *where* $m_1, m_2 \in \mathbb{N}$. *In addition, suppose that* $|K_1^c|^2 = |K_2^c|^2 = \sqrt{\sigma}$, *and suppose that* $\frac{\pi^2}{L_3^2} > \sqrt{\sigma}$. *Then* $\mathcal{S} = \{K_1^c, K_2^c\}$ *if and only if* $\gcd(m_1, m_2)$ *is not divisible by any primes congruent to 1 modulo 3.*

Proof. Suppose there exists $K_3^c = \left(\frac{k_1\pi}{L_1}, \frac{k_2\pi}{L_2}, \frac{k_3\pi}{L_3}\right)$ such that $|K_1^c|^2 = |K_2^c|^2 = |K_3^c|^2$. Since $\frac{\pi^2}{L_3^2} > \sqrt{\sigma}$, $k_3 = 0$. Therefore, since $|K_2^c|^2 = |K_3^c|^2$,

$$\frac{k_1^2\pi^2}{L_1^2} + \frac{k_2^2\pi^2}{L_2^2} = \frac{m_1^2\pi^2}{L_1^2} + \frac{m_2^2\pi^2}{L_2^2}.$$

Since $L_1 = \frac{m_1\sqrt{3}L_2}{m_2}$, we can recast this equation as

$$\frac{k_1^2}{m_1^2} + \frac{3k_2^2}{m_2^2} = 4.$$

We are therefore interested in determining the integer solutions to

$$\frac{x^2}{m_1^2} + \frac{3y^2}{m_2^2} = 4. \tag{9}$$

We can parameterize the solution curve using a line through the solution at $(2m_1, 0)$ of slope t. When we do so, we obtain

$$x = \frac{2m_1(3t^2m_1^2 - m_2^2)}{3t^2m_1^2 + m_2^2},$$
$$y = \frac{-4tm_1m_2^2}{3t^2m_1^2 + m_2^2}.$$

Since we have centered our parameterization at a rational solution to (9), we can only obtain rational solutions when t is itself rational. We let $t = \frac{am_2}{bm_1}$, where $\gcd(a, b) = 1$. We obtain

$$x = \frac{2m_1(3a^2 - b^2)}{3a^2 + b^2},$$
$$y = \frac{-4m_2ab}{3a^2 + b^2}.$$

Note that we can obtain all rational solutions up to symmetry across the x-axis by taking a and b to be nonnegative. Note that $(a, b) = (0, 1)$ produces the solution $(x, y) = (-2m_1, 0)$. $(a, b) = (1, 0)$ produces the solution $(x, y) = (2m_1, 0)$. $(a, b) = (1, 1)$ produces the solution $(x, y) = (m_1, -m_2)$. $(a, b) = (1, 3)$ produces the solution $(x, y) = (-m_1, -m_2)$. These solutions (and their reflections across the x-axis) are already accounted for by K_1^c and K_2^c. So in order for K_3^c to be different from K_1^c and K_2^c, it is both necessary and sufficient to prove the existence of $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$ such that $(a, b) \notin \{(0, 1), (1, 0), (1, 1), (1, 3)\}$, $3a^2 + b^2 \mid 2m_1 (3a^2 - b^2)$, and $3a^2 + b^2 \mid 4m_2ab$. The remainder of this proof is in the appendix that follows. □

3. Appendix A: Number Theoretic Details in the Proof of Proposition 2.6

In the proof of Proposition 2.6, we showed that there exists $K_3^c \in \mathcal{P} \setminus \{K_1^c, K_2^c\}$ satisfying $|K_1^c|^2 = |K_2^c|^2 = |K_3^c|^2$ if and only if there exist coprime nonnegative integers a and b with $(a, b) \notin \{(0, 1), (1, 0), (1, 1), (1, 3)\}$ such that

$$3a^2 + b^2 \mid 2m_1 \left(3a^2 - b^2\right), \quad 3a^2 + b^2 \mid 4m_2ab. \tag{10}$$

We now show that such a and b exist if and only if there exists a prime $p \equiv 1 \bmod 3$ such that $p \mid \gcd(m_1, m_2)$. To do so, we first must prove some preliminary results:

Lemma 3.1. *If* $\gcd(a, b) = 1$, *then* $\gcd\left(3a^2 + b^2, 3a^2 - b^2\right) = \gcd\left(3a^2 + b^2, 6\right)$.

Proof. We let $d_1 := \gcd(3a^2 + b^2, 3a^2 - b^2)$ and $d_2 := \gcd(3a^2 + b^2, 6)$. Then d_1 divides any integer linear combination of $3a^2 + b^2$ and $3a^2 - b^2$, meaning $d_1 \mid (3a^2 + b^2) + (3a^2 - b^2) = 6a^2$ and $d_1 \mid (3a^2 + b^2) - (3a^2 - b^2) = 2b^2 \mid 6b^2$. So $d_1 \mid \gcd(6a^2, 6b^2) = 6 \gcd(a^2, b^2) = 6$. Since $d_1 \mid 3a^2 + b^2$, we get that $d_1 \mid d_2$. Now note that since $d_2 \mid 6$, we have that $d_2 = 1$, $d_2 = 2$, $d_2 = 3$, or $d_2 = 6$. If $d_2 = 1$, then $d_2 \mid d_1$ trivially. If $d_2 = 2$, then $2 \mid 3a^2 + b^2$, meaning a and b must both be odd. But then $2 \mid 3a^2 - b^2$ as well, meaning $2 \mid d_1$, so $d_2 \mid d_1$. If $d_2 = 3$, then in order to have $3 \mid 3a^2 + b^2$, we must have $3 \mid b$. This means that $3 \mid 3a^2 - b^2$ as well. Thus, $3 \mid d_1$, so $d_2 \mid d_1$. Finally, if $d_2 = 6$, then a and b must both be odd so that $2 \mid 3a^2 + b^2$, and $3 \mid b$ so that $3 \mid 3a^2 + b^2$. As in the previous two cases, we deduce that $2 \mid 3a^2 - b^2$ and $3 \mid 3a^2 - b^2$. Therefore, $6 \mid 3a^2 - b^2$, meaning $6 \mid d_1$, i. e. $d_2 \mid d_1$. In all cases, we get that $d_2 \mid d_1$. Since $d_1 \mid d_2$ as well, this proves that $d_1 = d_2$. □

Lemma 3.2. *If* $\gcd(a, b) = 1$*, then* $\gcd(3a^2 + b^2, ab) = \gcd(3a^2 + b^2, 3)$.

Proof. We let $d_1 := \gcd(3a^2 + b^2, ab)$ and $d_2 := \gcd(3a^2 + b^2, 3)$. We let $q \neq 3$ be a prime such that $q \mid 3a^2 + b^2$. Since a and b cannot have q as a common factor, we must have $\gcd(a, q) = \gcd(b, q) = 1$. But then q cannot divide ab. Also note that if $9 \mid 3a^2 + b^2$, then we must have $3 \mid b$ so that $3 \mid 3a^2 + b^2$. But then we would need $3 \mid a$ to guarantee the divisibility by 9, which is impossible since $\gcd(a, b) = 1$. This means that the only possible common divisors of $3a^2 + b^2$ and ab are 1 and 3, which means that $d_1 \mid 3$. Thus, $d_1 \mid d_2$. Now note that since $d_2 \mid 3$, we must have $d_2 = 1$ or $d_2 = 3$. If $d_2 = 1$, then $d_2 \mid d_1$ trivially. If $d_2 = 3$, then $3 \mid 3a^2 + b^2$, meaning $3 \mid b$. Thus, $3 \mid ab$ as well, which implies that $d_2 \mid d_1$. Therefore, $d_1 = d_2$. □

Lemma 3.3. *Suppose that* $\gcd(a, b) = 1$*. Then* $3a^2 + b^2 \mid 2m_1(3a^2 - b^2)$ *if and only if* $3a^2 + b^2 \mid 12m_1$.

Proof. To prove the forward direction, we assume that $3a^2 + b^2 \mid 2m_1(3a^2 - b^2)$. By Lemma 3.1, this means that

$$\begin{aligned}
3a^2 + b^2 &\mid \gcd\left(3a^2 + b^2, 2m_1\left(3a^2 - b^2\right)\right) \\
&\mid \gcd\left(3a^2 + b^2, 2m_1\right) \gcd\left(3a^2 + b^2, 3a^2 - b^2\right) \\
&= \gcd\left(3a^2 + b^2, 2m_1\right) \gcd\left(3a^2 + b^2, 6\right) \\
&\mid 2m_1(6) \\
&= 12m_1.
\end{aligned}$$

To prove the reverse direction, we assume that $3a^2 + b^2 \mid 12m_1$. By Lemma 3.1, this means that

$$\begin{aligned}
3a^2 + b^2 &\mid \gcd\left(3a^2 + b^2, 12m_1\right) \\
&\mid \gcd\left(3a^2 + b^2, 2m_1\right) \gcd\left(3a^2 + b^2, 6\right) \\
&= \gcd\left(3a^2 + b^2, 2m_1\right) \gcd\left(3a^2 + b^2, 3a^2 - b^2\right) \\
&\mid 2m_1(3a^2 - b^2).
\end{aligned}$$

□

Lemma 3.4. *Suppose that* $\gcd(a, b) = 1$*. Then* $3a^2 + b^2 \mid 4m_2ab$ *if and only if* $3a^2 + b^2 \mid 12m_2$.

Proof. To prove the forward direction, we assume that $3a^2 + b^2 \mid 4m_2ab$. By Lemma 3.2, this means that

$$\begin{aligned}
3a^2 + b^2 &\mid \gcd\left(3a^2 + b^2, 4m_2ab\right) \\
&\mid \gcd\left(3a^2 + b^2, 4m_2\right) \gcd\left(3a^2 + b^2, ab\right) \\
&= \gcd\left(3a^2 + b^2, 4m_2\right) \gcd\left(3a^2 + b^2, 3\right) \\
&\mid 4m_2(3) \\
&= 12m_2.
\end{aligned}$$

To prove the reverse direction, we assume that $3a^2 + b^2 \mid 12m_2$. By Lemma 3.2, this means that

$$3a^2 + b^2 \mid \gcd\left(3a^2 + b^2, 12m_2\right)$$

$$\begin{aligned}&\mid \gcd\left(3a^2+b^2,4m_2\right)\gcd\left(3a^2+b^2,3\right)\\&=\gcd\left(3a^2+b^2,4m_2\right)\gcd\left(3a^2+b^2,ab\right)\\&\mid 4m_2ab.\end{aligned}$$ □

The previous two results imply that (10) is equivalent to

$$3a^2+b^2 \mid 12\gcd(m_1,m_2).$$

We now provide some results on which positive integers cannot be expressed as $3a^2+b^2$ for $a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$.

Lemma 3.5. *Suppose* $n\equiv 0 \bmod 8$. *Then* n *cannot be expressed as* $3a^2+b^2$ *for* $a,b\in\mathbb{Z}$ *with* $\gcd(a,b)=1$.

Proof. Suppose towards contradiction that there exist $a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$ such that $n=3a^2+b^2$. Note that a and b cannot both be even since they are coprime, so a and b must both be odd because n is even. But then $a^2\equiv b^2\equiv 1 \bmod 8$, meaning $n\equiv 4 \bmod 8$, a contradiction. □

Lemma 3.6. *Suppose* $n\equiv 2 \bmod 4$. *Then* n *cannot be expressed as* $3a^2+b^2$ *for* $a,b\in\mathbb{Z}$ *with* $\gcd(a,b)=1$.

Proof. Suppose towards contradiction that there exist $a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$ such that $n=3a^2+b^2$. Since n is even, we must have that $a\equiv b\equiv 1 \bmod 2$ by the same reasoning as in the proof of the previous result. Therefore, $a^2\equiv b^2\equiv 1 \bmod 4$, which implies that $n\equiv 0 \bmod 4$. This is a contradiction. □

Lemma 3.7. *Suppose* $n\equiv 0 \bmod 9$. *Then* n *cannot be expressed as* $3a^2+b^2$ *for* $a,b\in\mathbb{Z}$ *with* $\gcd(a,b)=1$.

Proof. Suppose towards contradiction that there exist $a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$ such that $n=3a^2+b^2$. If $n\equiv 0 \bmod 9$, then $n\equiv 0 \bmod 3$ as well, which means that $b\equiv 0 \bmod 3$. But since $n\equiv 0 \bmod 9$, we must have $a\equiv 0 \bmod 3$. This is a contradiction because a and b cannot have 3 as a common divisor. □

Lemma 3.8. *Let* $p\geq 5$ *be prime. Then* -3 *is a quadratic residue modulo 3 if and only if* $p\equiv 1 \bmod 3$.

Proof. Using Legendre symbols, we have

$$\left(\frac{-3}{p}\right)=\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)=(-1)^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}}\left(\frac{p}{3}\right)=(-1)^{p-1}\left(\frac{p}{3}\right)=\left(\frac{p}{3}\right).$$

Now note that since 1 is a quadratic residue modulo 3 and 2 is not, $\left(\frac{-3}{p}\right)=1$ if $p\equiv 1 \bmod 3$, and $\left(\frac{-3}{p}\right)=-1$ if $p\equiv 2 \bmod 3$. This proves the claim. □

Lemma 3.9. *Suppose* $n\equiv 0 \bmod p$, *where* $p\geq 5$ *is a prime satisfying* $p\equiv 2 \bmod 3$. *Then* n *cannot be expressed as* $3a^2+b^2$ *for* $a,b\in\mathbb{Z}$ *with* $\gcd(a,b)=1$.

Proof. Suppose towards contradiction that there exist $a,b\in\mathbb{Z}$ with $\gcd(a,b)=1$ such that $n=3a^2+b^2$. Thus, $3a^2+b^2\equiv 0 \bmod p$. Note that if $a\equiv 0 \bmod p$, then $b\equiv 0 \bmod p$ as well, which is impossible since $\gcd(a,b)=1$. Thus, $a\not\equiv 0 \bmod p$, which means a has a multiplicative inverse modulo p. Therefore, we have that $-3\equiv \left(a^{-1}b\right)^2 \bmod p$. Therefore, -3

is a quadratic residue modulo p. So by the previous lemma, $p \equiv 1 \bmod 3$, which contradicts our assumption that $p \equiv 2 \bmod 3$. □

Our next lemma is a theorem of Fermat, which is proven in the exercises in [5].

Lemma 3.10. *Let* $p \equiv 1 \bmod 3$ *be prime. Then there exist* $a, b \in \mathbb{Z}$ *with* $\gcd(a,b) = 1$ *such that* $p = 3a^2 + b^2$.

With these tools, we can now complete the proof of Proposition 2.6. We know that there exists $K_3^c \in \mathcal{P} \setminus \{K_1^c, K_2^c\}$ satisfying $|K_1^c|^2 = |K_2^c|^2 = |K_3^c|^2$ if and only if there exist coprime nonnegative integers a and b with $(a,b) \notin \{(0,1),(1,0),(1,1),(1,3)\}$ such that $3a^2 + b^2 \mid 12\gcd(m_1,m_2)$. We first prove the reverse direction by supposing there exists a prime $p \equiv 1 \bmod 3$ such that $p \mid \gcd(m_1,m_2)$. Then by Lemma 3.10, $p = 3a^2 + b^2$ for some $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$. By extension, $3a^2 + b^2 \mid 12\gcd(m_1,m_2)$. Also, $(a,b) \notin \{(0,1),(1,0),(1,1),(1,3)\}$ since $3a^2+b^2$ does not equal a prime congruent to 1 modulo 3 in any of these cases. Thus, there exists $K_3^c \in \mathcal{P} \setminus \{K_1^c, K_2^c\}$ satisfying $|K_1^c|^2 = |K_2^c|^2 = |K_3^c|^2$ in this case. Conversely, if no such p exists, then $12\gcd(m_1,m_2)$ factors into a product of 2s, 3s, and odd primes congruent to 2 modulo 3. But then Lemma 3.5, Lemma 3.6, Lemma 3.7, and Lemma 3.9 imply that the only divisors of $12\gcd(m_1,m_2)$ that can be expressed as $3a^2 + b^2$ for $a, b \in \mathbb{Z}$ with $\gcd(a,b) = 1$ are 1, 3, 4 and 12. However, these correspond to $(a,b) \in \{(0,1),(1,0),(1,1),(1,3)\}$. Thus, there does not exist $K_3^c \in \mathcal{P} \setminus \{K_1^c, K_2^c\}$ satisfying $|K_1^c|^2 = |K_2^c|^2 = |K_3^c|^2$ in this case. This proves the claim.

4. Appendix B: Maple Calculations

On the following pages, we include Maple calculations for the integral in (4).

> $restart;$

> $e1 := \cos\left(\frac{2\cdot m1\cdot \mathrm{Pi}\cdot x1}{L1}\right); e2 := \cos\left(\frac{m1\cdot \mathrm{Pi}\cdot x1}{L1}\right)\cdot\cos\left(\frac{m2\cdot \mathrm{Pi}\cdot x2}{L2}\right);$

$$e1 := \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right)$$

$$e2 := \cos\left(\frac{m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) \tag{1}$$

> $u := y1\cdot e1 + y2\cdot e2 + \mathrm{Phi}$

$$u := \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) y1 + \cos\left(\frac{m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) y2 + \Phi \tag{2}$$

> **local** D; $B := 0$; D $:= 0$

Warning, A new binding for the name \`D\` has been created. The global instance of this name is still accessible using the :- prefix, :-\`D\`. See ?protect for details.

$$\mathrm{D}$$

$$B := 0$$

$$\mathrm{D} := 0 \tag{3}$$

> $\mathrm{Phi} := -A\cdot y1^2\cdot\cos\left(\frac{4\cdot m1\cdot \mathrm{Pi}\cdot x1}{L1}\right) - B\cdot y1\cdot y2\cdot\cos\left(\frac{m1\cdot \mathrm{Pi}\cdot x1}{L1}\right)\cdot\cos\left(\frac{m2\cdot \mathrm{Pi}\cdot x2}{L2}\right) - C\cdot y1\cdot y2 \cdot\cos\left(\frac{3\cdot m1\cdot \mathrm{Pi}\cdot x1}{L1}\right)\cdot\cos\left(\frac{m2\cdot \mathrm{Pi}\cdot x2}{L2}\right) - \mathrm{D}\cdot y2^2\cdot\cos\left(\frac{2\cdot m1\cdot \mathrm{Pi}\cdot x1}{L1}\right) - E\cdot y2^2 \cdot\cos\left(\frac{2\cdot m2\cdot \mathrm{Pi}\cdot x2}{L2}\right) - F\cdot y2^2\cdot\cos\left(\frac{2\cdot m1\cdot \mathrm{Pi}\cdot x1}{L1}\right)\cdot\cos\left(\frac{2\cdot m2\cdot \mathrm{Pi}\cdot x2}{L2}\right)$

$$\Phi := -A\, y1^2 \cos\left(\frac{4\, m1\, \pi\, x1}{L1}\right) - C\, y1\, y2 \cos\left(\frac{3\, m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) - E\, y2^2 \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right) - F\, y2^2 \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right)\cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right) \tag{4}$$

> $f := \left(gamma2\cdot u^2 + gamma3\cdot u^3\right)\cdot e1; g := \left(gamma2\cdot u^2 + gamma3\cdot u^3\right)\cdot e2$

$$f := \Bigg(\gamma3\,\Bigg(\cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) y1 + \cos\left(\frac{m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) y2 - A\, y1^2\cos\left(\frac{4\, m1\, \pi\, x1}{L1}\right) - C\, y1\, y2\cos\left(\frac{3\, m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) - E\, y2^2\cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right) - F\, y2^2\cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right)\cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)\Bigg)^3 + \gamma2\,\Bigg(\cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) y1 + \cos\left(\frac{m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) y2 - A\, y1^2\cos\left(\frac{4\, m1\, \pi\, x1}{L1}\right) - C\, y1\, y2\cos\left(\frac{3\, m1\, \pi\, x1}{L1}\right)\cos\left(\frac{m2\, \pi\, x2}{L2}\right) - E\, y2^2\cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)$$

$$- F\, y2^2 \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)\Bigg)^2\Bigg) \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right)$$

$$g := \Bigg(\gamma 3 \Bigg(\cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) y1 + \cos\left(\frac{m1\, \pi\, x1}{L1}\right) \cos\left(\frac{m2\, \pi\, x2}{L2}\right) y2 - A\, y1^2 \cos\left(\frac{4\, m1\, \pi\, x1}{L1}\right)$$ **(5)**

$$- C\, y1\, y2 \cos\left(\frac{3\, m1\, \pi\, x1}{L1}\right) \cos\left(\frac{m2\, \pi\, x2}{L2}\right) - E\, y2^2 \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)$$

$$- F\, y2^2 \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)\Bigg)^3 + \gamma 2 \Bigg(\cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) y1$$

$$+ \cos\left(\frac{m1\, \pi\, x1}{L1}\right) \cos\left(\frac{m2\, \pi\, x2}{L2}\right) y2 - A\, y1^2 \cos\left(\frac{4\, m1\, \pi\, x1}{L1}\right)$$

$$- C\, y1\, y2 \cos\left(\frac{3\, m1\, \pi\, x1}{L1}\right) \cos\left(\frac{m2\, \pi\, x2}{L2}\right) - E\, y2^2 \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)$$

$$- F\, y2^2 \cos\left(\frac{2\, m1\, \pi\, x1}{L1}\right) \cos\left(\frac{2\, m2\, \pi\, x2}{L2}\right)\Bigg)^2\Bigg) \cos\left(\frac{m1\, \pi\, x1}{L1}\right) \cos\left(\frac{m2\, \pi\, x2}{L2}\right)$$

> $rhs1 := collect\left(simplify\left(\frac{int(f, x1 = 0 ..L1, x2 = 0 ..L2, x3 = 0 ..L3)}{L1 \cdot L2 \cdot L3}\right), [y1, y2]\right)$ assuming $m1$ $:: integer, m2 :: integer$

$$rhs1 := \frac{3\, A^2\, \gamma 3\, y1^5}{4} - \frac{3\, A\, C^2\, \gamma 3\, y1^4\, y2^2}{16} + \left(\frac{3\, C\, (A + C)\, \gamma 3\, y2^2}{8} + \frac{3\, \gamma 3}{8} - \frac{A\, \gamma 2}{2}\right) y1^3$$ **(6)**

$$+ \left(\frac{\left(-12\, A\, E\, F - 3\, C^2\, F\right) \gamma 3\, y2^4}{16} - \frac{3\, (A + 2\, C)\, \gamma 3\, y2^2}{16}\right) y1^2$$

$$+ \left(\frac{\left((6\, E + 3\, F)\, C + 12\, E^2 + 9\, F^2\right) \gamma 3\, y2^4}{16} + \left(-\frac{C\, \gamma 2}{4} + \frac{3\, \gamma 3}{8}\right) y2^2\right) y1$$

$$+ \left(\frac{(-3\, E - 3\, F)\, \gamma 3}{16} + \frac{F\, E\, \gamma 2}{2}\right) y2^4 + \frac{\gamma 2\, y2^2}{8}$$

> $rhs2 := collect\left(simplify\left(\frac{int(g, x1 = 0 ..L1, x2 = 0 ..L2, x3 = 0 ..L3)}{L1 \cdot L2 \cdot L3}\right), [y2, y1]\right)$ assuming $m1$ $:: integer, m2 :: integer$

$$rhs2 := \left(-\frac{3\, C\, F\left(E + \frac{F}{4}\right) \gamma 3\, y1}{8} + \frac{3\left(E^2 + \frac{1}{2}\, F^2 + E\, F\right) \gamma 3}{8}\right) y2^5 + \Bigg($$ **(7)**

$$-\frac{3\left(E + \frac{F}{2}\right) C\, A\, \gamma 3\, y1^3}{8} + \frac{3\left(\frac{3\, C^2}{4} + \left(E + \frac{F}{2}\right) C + \frac{A\, F}{2}\right) \gamma 3\, y1^2}{8}$$

$$+ \left(\frac{3\left(-\frac{3\, C}{8} - E - F\right) \gamma 3}{8} + \frac{C\, F\, \gamma 2}{8}\right) y1 + \frac{9\, \gamma 3}{64} + \frac{\gamma 2\left(-E - \frac{F}{2}\right)}{4}\Bigg) y2^3$$

$$+\left(\frac{3\,A\,(A+C)\,\gamma 3\,y1^4}{8}+\left(\frac{3\left(-A-\frac{C}{2}\right)\gamma 3}{8}+\frac{A\,C\,\gamma 2}{4}\right)y1^3+\left(-\frac{C\,\gamma 2}{4}+\frac{3\,\gamma 3}{8}\right)y1^2\right.$$

$$\left.+\frac{\gamma 2\,y1}{4}\right)y2$$

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