TRANSITION DENSITY OF BROWNIAN MOTIONS ON METRIC GRAPHS

JUSTIN LIU

Abstract. In this report, I will give a brief overview of the construction of Brownian motion on metric graphs, and then outline known results and estimates for the transition density of Brownian motions on metric graphs, with a focus firstly on compact metric graphs and then infinite symmetric metric trees.

1. Introduction

Brownian motions are continuous time stochastic processes, first used to model the random motion of particles suspended in a medium, named after the botanist Robert Brown who described the phenomenon in 1827, and made formal by the work of Bachelier and Einstein. Today, Brownian motion is used in models in wide-ranging applications in fields like physics, finance, biology and more.

Metric graphs have also been growing in popularity as models in many different domains ([Pos07], [FS00]). Notably, there has been notable recent research on solving stochastic partial differential equations on metric graphs ([FAN21], [HR14]). Much recent work has been also done on infinite metric trees due to their inherent symmetrical structure ([FHT21]), which will be a focus of one of the sections in this report.

In Section 2, I summarize preliminary definitions and elementary examples. Then, in the following sections. I summarize the bijection between Brownian motion, semigroup operators and Dirichlet forms, in order to formally construct Brownian motion on metric graphs. Finally, I will first summarize known results for compact metric graphs, and then I will summarize known results for (infinite) metric trees.

2. Preliminaries

Definition 2.1. *An $\mathbb{R}^d$-valued stochastic process $B(t) : t \geq 0$ is called a d-dimensional Brownian motion starting at $\mathbf{x} \in \mathbb{R}^d$ if:*

1. $B(0) = \mathbf{x}$.
2. *For all $0 \leq t_1 \leq \cdots \leq t_n$, the increments $B(t_n) - B(t_{n-1}), \ldots, B(t_2) - B(t_1)$ are independent random variables.*
3. *For all $t \geq 0$ and $h > 0$, the increment $B(t+h) - B(t)$ is distributed normally with a mean of 0 and variance of h.*
4. *Almost surely, the function $t \mapsto B(t)$ is continuous.*

This report primarily concerns 1-dimensional Brownian motions starting at 0. This will be referred to as Standard Brownian Motion.

The primary space we will be working in is a metric graph $\Gamma = (V, E)$, where each $e \in E$ is associated with a closed interval $[0, l(e)]$ if $l(e) < \infty$, or $[0, \infty)$. The goal of this report is to summarize known results about the transition density of Brownian motions on metric graphs.

The simplest example of a metric graph is the non-negative real line $\mathbb{R}_+$, which is a metric graph with one vertex and one edge with infinite length. This is analogous to Reflected Brownian Motion:

Theorem 1. *Let $W(t)$ be a Brownian motion on $\mathbb{R}_+$ and let $B(t)$ denote Standard Brownian Motion. Denote the transition density function of $W(t)$ by p_W and similarly the transition density of $B(t)$ by p_B. Then, we have:*

$$p_W(t,x,y) = p_B(t,x,y) + p_B(t,x,-y) \tag{1}$$

$$= \frac{1}{\sqrt{2\pi t}}\left(e^{\frac{(y-x)^2}{2t}} + e^{\frac{(y+x)^2}{2t}}\right). \tag{2}$$

Now, consider Brownian motion on a bounded interval $[0,a]$. Using a similar principle, we can see that:

Theorem 2. *Let $W(t)$ be a Brownian motion on $[0,a]$ and let $B(t)$ denote Standard Brownian Motion. Denote the transition density function of $W(t)$ by p_W and similarly the transition density of $B(t)$ by p_B. Then, we have:*

$$p_W(t,x,y) = \sum_{n=-\infty}^{\infty} p_B(t,x,y+2na). \tag{3}$$

Unfortunately, if we wish to study Brownian motions on metric graphs in general, we are no longer able to find nice formulas for transition densities in terms of Standard Brownian Motion.

3. Constructing Brownian Motion on Metric Graphs

3.1. **The Heat Kernel.** Consider the Standard Brownian Motion $B(t)$. The transition density function is equal to the normal probability density function:

$$p(t,0,x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

We can notice that

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} p(t,0,x) = \frac{\partial}{\partial t} p(t,0,x).$$

This leads us to the interesting connection between Brownian motions and the heat kernel.

Theorem 3. *Let u be a solution to the heat equation. In other words:*

$$\frac{\partial}{\partial t} u(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(x,t),$$

with initial condition $u(x,0) = f(x)$ for some function f.

Informally, $u(t,x)$ represents the density of heat particles at position x at time t. So, if we imagine there are $f(y)$ heat particles at y at time 0 and the fraction that are at x is the probability that a particle at y moved to x, we can reformulate the problem:

$$u(t,x) = \int_I f(y) p(t,y,x)\mathrm{d}y \tag{4}$$

$$= \int_I f(y) p(t,x,y)\mathrm{d}y \tag{5}$$

$$= \mathbb{E}[f(W(t)) : W(0) = x], \tag{6}$$

where I is the length of the 1-dimensional "rod" and $W(t)$ is a Brownian motion on I.

3.2. **Dirichlet Forms.** Another way to construct Brownian motions on metric graphs is by considering Dirichlet forms.

Definition 3.1. *A Dirichlet form on a measure space $(X, \mathcal{A}, \mu)$ is a bilinear function $\epsilon : D \times D \to \mathbb{R}$ such that:*

1. *$D \subseteq L^2(\mu)$ is dense.*
2. *ϵ is symmetric; that is, $\epsilon(f, g) = \epsilon(g, f)$.*
3. *$\epsilon(u, u) \geq 0$ for all $u \in D$.*
4. *D equipped with the inner product $(u, v)_\epsilon := (u, v)_{L^2(\mu)} + \epsilon(u, v)$ is a real Hilbert space.*
5. *For all $u \in D$, $u_* = \min(\max(u, 0), 1) \in D$ and $\epsilon(u_*, u_*) \leq \epsilon(u, u)$.*

Consider a metric graph $\Gamma = (V, E)$, and denote the length of each $e \in E$ by $\ell(e)$. Also consider associating to each edge e values $p(e)$ representing edge weight and $\omega(e)$ representing jump conductance. Consider the Hilbert space $L^2((\Gamma, \ell, p, \omega), \mu)$, where

$$\mu(\mathrm{d}x) = \sum_{e \in E} \mathbf{1}_{I(e)}(x) m(\mathrm{d}x),$$

where $I(e)$ is the interval associated with e and m is the Lebesgue measure on $I(e)$.

Let $C_c((\Gamma, \ell, p, \omega))$ be the set of compactly supported continuous functions on Γ, and $C_0((\Gamma, \ell, p, \omega))$ the closure of $C_c((\Gamma, \ell, p, \omega))$ with respect to the $\|\|_\infty$-norm.

Now, let $W^{k,p}(I(e), \mu|_{I(e)}$ denote the Sobolev space of functions in $L^p(I(e))$ such that its weak derivatives up to order k have finite L^p norm. Set for $k \in \mathbb{R}_+$ and $p \geq 1$,

$$S^{k,p}((\Gamma, \ell, p, \omega), \mu) := \{u \in C((\Gamma, \ell, p, \omega)) : \forall e \in E, u|_{I(e)} \in W^{k,p}(I(e), \mu|_{I(e)})\} \tag{7}$$

$$W^{k,p}((\Gamma, \ell, p, \omega), \mu) := \{u \in S^{k,p}((\Gamma, \ell, p, \omega), \mu) : \sum_{e \in E} \|u\|^p_{W_{k,p}(I(e), \mu|_{I_e}}) < \infty\} \tag{8}$$

$$W_0^{k,p}((\Gamma, \ell, p, \omega), \mu) := C_0((\Gamma, \ell, p, \omega)) \cap W^{k,p}((\Gamma, \ell, p, \omega), \mu). \tag{9}$$

Finally, consider the Dirichlet form on $L^2((\Gamma, \ell, p, \omega), \mu)$ with domain $W_0^{1,2}((\Gamma, \ell, p, \omega), \mu)$, defined by

$$\epsilon(f, g) = \sum_{e \in E} \int_{I(e)} f'(x) g'(x) p(e) m(\mathrm{d}x).$$

It is a well-established fact that every regular Dirichlet form has an associated Markov process [CF11, Theorem 1.5.1]. The process associated with ϵ is in fact a Brownian motion on $(\Gamma, \ell, p, \omega)$.[Fol14]

Informally, we can view this process as obtained from Brownian Motion on $\mathbb{R}$, denoted $B(t)$. If we start at some vertex x, then $\{B(t) \neq 0\}$ consists of countably many intervals. Now, during each of these intervals, we pick an adjacent edge f with probability $\frac{p(f)}{\sum_{e \in E(x)} p(e)}$, where $E(x)$ denotes all edges that connect to x, and the process moves like $\left|W_{\omega(f)^{-1}t}\right|$ until it reaches a new vertex, at which point, we can use the strong Markov property and continue.

4. Well-Known Results about Brownian Motions on Metric Graphs

We denote Brownian motion on a metric graph Γ as $(Y_t)_{t \geq 0}$.

Pick a vertex $x \in \Gamma$ that is adjacent to vertices $x_1, \ldots, x_k$ by edges $e_1, \ldots, e_k$, and itself through the edge x_{loop}. Let B be the part of the graph that contains these.

We also introduce a coordinate system: for $1 \leq j \leq k$ and $0 \leq z \leq \ell(e_j)$, we call (j, z) the point on e_j that is at distance z from x. On the loop, we choose some arbitrary orientation and write $(loop, z)$.

Finally, let $\tau = \inf\{t \geq 0 : Y_t \notin B\}$, which is the hitting time for $x_1, \ldots, x_k$.

Theorem 4. *Given a Brownian motion (Y_t) starting at $x \in \Gamma$, the probability that Y_t hits the vertex e_j first is*

$$\mathbb{P}^x(Y_\tau = x_j) = \frac{q(e_j)/\ell(e_j)}{\sum_{e \in E(x)} q(e)/\ell(e)}, \tag{10}$$

where

$$q(e_j) = \begin{cases} p(e_j) & \text{if } e \neq loop \\ 2p(e_j) & \text{if } e = loop \end{cases}, \tag{11}$$

and $E(x)$ denotes all edges that connect to x.[Fol14, Theorem 2.1]

5. Compact Metric Graphs

The heat kernel on compact metric graphs is very well understood, and we have a lot of nice methods for calculating the heat kernel.

5.1. Known Results.

Let Γ be a compact metric graph. On each edge, the Laplacian $-\Delta$ is subject to Kirchhoff-Neumann boundary conditions. We associate with $-\Delta$ an orthonormal basis of real-valued eigenfunctions ψ_j with corresponding eigenvalues λ_j. Then, it is well-known that the heat kernel can be calculated using this formula:[BHJ22]

$$H(t, x, y) = \sum_{j=1}^{\infty} e^{-t\lambda_j} \psi_j(x)\psi_j(y). \tag{12}$$

Example 1. *Let Γ be a star graph with d equal edges of length a. Then, considering an orthonormal basis of eigenfunctions for the Laplacian on Γ, using the above formula we can find the expansion:*

$$H(t, x, x) = \frac{1}{ad} + \frac{2}{ad} \sum_{k=1}^{\infty} e^{(-\pi k/a)^2 t} \cos^2(\frac{\pi k}{a} x) + \frac{2(d-1)}{ad} \sum_{k=0}^{\infty} e^{(\frac{\pi}{a}(k+\frac{1}{2})^2 t} \sin^2(\frac{\pi}{a}(k + \frac{1}{2})x). \tag{13}$$

5.2. Setup.

Let Γ be a compact metric graph, meaning that the number of edges and vertices is finite, with each edge length being finite as well. Since Γ is an "undirected" graph, we can turn it into a directed graph by turning each edge into two directed edges (which will henceforth be referred to as bonds), with one in each direction.

For each bond $\vec{e}$, we denote the initial vertex as $\partial^-(\vec{e})$ and the final vertex as $\partial^+(\vec{e})$, and we say that two bonds $\vec{e_1}$ and $\vec{e_2}$ are consecutive if $\partial^+(\vec{e_2}) = \partial^-(\vec{e_1})$. Given two consecutive bonds $\vec{e_1}$ and $\vec{e_2}$, we say that they form a *transfer* if $\partial^-(\vec{e_1}) \neq \partial^+(\vec{e_2})$, and a *bounce* if $\partial^-(\vec{e_1}) = \partial^+(\vec{e_2})$.

Using consecutive bonds, we can build a path between two vertices v_1 and v_2, denoted $\gamma = (v_1, \vec{e_1}, \ldots, \vec{e_n}, v_2)$, where $\partial^-(\vec{e_1}) = v_1$ and $\partial^+(\vec{e_n}) = v_2$, and we denote the length of the path as $\ell(\gamma)$.

Finally, let $\mathcal{P}(v_1, v_2)$ be the collection of all paths from v_1 to v_2, and to each path γ we assign a coefficient $\alpha(\gamma)$, defined as follows:

$$\alpha((v,v)) := \frac{2}{\deg(v)}, \tag{14}$$

where (v, v) is the trivial path from a vertex v to itself (note that the path does not travel through any bonds).

Now, for a general path, we define:

$$\alpha((v_-, \vec{e_1}, \ldots, \vec{e_n}, v_+)) := \frac{4}{\deg(v_-)\deg(v_+)} \prod_{j=1}^{n} \beta(\vec{e_j}, \vec{e_{j+1}}), \tag{15}$$

where

$$\beta(\vec{e_j}, \vec{e_{j+1}}) := \begin{cases} \frac{2}{\deg \partial^+(\vec{e_j})}, & \partial^-(\vec{e_j}) \neq \partial^+(\vec{e_{j+1}}) \text{ (transfer)} \\ \frac{2}{\deg \partial^+(\vec{e_j})} - 1, & \partial^-(\vec{e_j} = \partial^+(\vec{e_{j+1}}) \text{ (bounce)} \end{cases} \tag{16}$$

An important distinction to be made is that we do not want to allow for tadpoles, which are bonds with equal initial and final vertices. We can work around this issue by inserting an artificial vertex of degree 2:

(A) A tadpole (B) No longer a tadpole

FIGURE 1. Tadpole: Before and After

These artificial vertices contribute a factor of 0 for a bounce and 1 for a transfer, making them essentially invisible except as potential terminal points for a path in the calculation of the coefficient of a path.

5.3. A Path Theorem for Compact Metric Graphs.

Theorem 5. *Given a compact metric graph Γ, and given two points $q_1, q_2 \in \Gamma$ (including vertices), the heat kernel admits the expansion*

$$H(t, q_1, q_2) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in \mathcal{P}(q_1,q_2)} \alpha(\gamma) e^{-\ell(\gamma)^2/4t}, \tag{17}$$

where $\alpha(\gamma)$ denotes the coefficient of the path γ, $\ell(\gamma)$ denotes the length, and $\mathcal{P}(q_1, q_2)$ is the collection of paths from q_1 to q_2.

5.4. **Path Formula for One-Dimensional Finite Interval.** Let Γ consist of two vertices and an edge of finite length $a > 0$. From here on out, we will simply regard Γ as being equivalent to $[0, a]$. Our goal is to use Theorem 5 to find a closed form expansion for the heat kernel on Γ. In other words, we pick x, y in the interior of Γ (i.e. not the endpoints of the interval), where without loss of generality $x < y$, and we wish to find $H(t, x, y)$.

First, we can make some observations:

(1) A potential path between x and y **cannot** bounce at x or y. This is because x and y, if we consider them as "artificial" nodes in the graph, have degree 2, and will any such path will involve some paths $\vec{e_j}$ and $\vec{e_{j+1}}$ such that $\beta(\vec{e_j}, \vec{e_{j+1}}) = 0$, and therefore the coefficient of that path will be 0, thus not counting towards the formula.

(2) However, a path **can** bounce at the 0 or a vertices which have degree 1.

Given these observations, we first begin by considering the most rudimentary "types" of paths. The two most basic types of paths are:

(1) A path going straight from x to y (henceforth denoted path type A).

(2) A path going from x to 0 to y (henceforth denoted path type B).

These are the shortest possible "allowed" paths that will count towards the summation.

(A) Path Type A (B) Path Type B

FIGURE 2. Two Rudimentary Path Types

Now, given these two path types, we can construct all possible paths by starting with one of the rudimentary types, then allowing the particle to bounce at 0 or 1 as many times as desired, as long as the path does not bounce at x or y.

(3) Starting with path type A, bouncing between 0 and a, then approaching y from the right. The length of such a path would be $|x + y + 2ak|$ for some $k \in -\mathbb{N}$.

(4) Starting with path type A, bouncing between 0 and a, then approaching y from the left. The length of such a path would be $|x - y + 2ak|$, for some $k \in -\mathbb{N}$.

(5) Starting with path type B, bouncing between 0 and a, then approaching y from the right. The length of such a path would be $|x - y + 2ak|$ for some $k \in \mathbb{N}$.

(6) Starting with path type B, bouncing between 0 and a, then approaching y from the left. The length of such a path would be $|x + y + 2ak|$ for some $k \in \mathbb{N}$.

Path Type (3) Path Type (5)

FIGURE 3. Examples of Constructed Paths

Therefore, we have that $\mathcal{P}(x, y)$ consists exactly of all the 6 above types of paths for every $k \in \mathbb{N}$. We can also see that for all $\gamma \in \mathcal{P}(x, y)$, $\alpha(\gamma) = 1$.

Therefore, plugging everything into the path summation formula, we get:

$$H(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{k\in\mathbb{R}} \left(e^{-|x-y+2ak|^2/4t} + e^{-|x+y+2ak|^2/4t} \right). \tag{18}$$

5.5. **Solving the Heat Equation on One-Dimensional Finite Interval.** Another way to obtain a formula for the heat kernel on $[0,a]$ is to directly solve the heat equation given Neumann boundary conditions at 0 and a. In other words, we want to solve for the function $H(t,x,y)$ such that for $u(t,x) := \int_0^a \phi(y)H(t,x,y)\mathrm{d}y$:

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ u(x,0) = \phi(x) \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(a,t) = 0 \end{cases}. \tag{19}$$

In order to solve this, we use separation of variables and find a solution of the form $u(x,t) = X(x)T(t)$. Plugging this expression, we get

$$\begin{aligned} XT' - kX''T &= 0 \\ \Rightarrow \frac{T'}{kT} = \frac{X''}{X} &=: -\lambda. \end{aligned}$$

So, the problem reduces to finding X and λ such that:

$$\begin{cases} -X''(x) = \lambda X(x) \\ X'(0) = X'(a) = 0 \end{cases}. \tag{20}$$

Solving this differential equation, we can see that general solutions are given by

$$X(x) = a_n \cos\left(\frac{n\pi}{a}\right). \tag{21}$$

Therefore, the solution is

$$u(x,t) = a_0 u_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos \mu_n x, \tag{22}$$

for any choice of $a_1, a_2, \ldots$, where $\mu_n = \frac{n\pi}{a}$ and $\lambda_n = \sqrt{k}\mu_n$.

Now, we solve for the initial condition $u(x,0) = \phi(x)$ to get that

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right), \tag{23}$$

for $x \in (0,a)$. This is a Fourier cosine series, and so we can write

$$\begin{aligned} a_0 &= \frac{1}{a}\int_0^a \phi(x)\mathrm{d}x, \\ a_n &= \frac{2}{a}\int_0^a \phi(x)\cos\left(\frac{n\pi x}{a}\right). \end{aligned}$$

By the linearity of integrals and plugging in appropriate values, we recover the formula

$$H(t,x,y) = \frac{1}{a} + \frac{2}{a}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi n}{a}\right)^2 t} \cos^2\left(\frac{n\pi x}{a}\right). \tag{24}$$

This matches up with (13). It is also important to note that if we solve this system with the method of images, we get the path formula (18).

5.6. **Other Remarks.** The path formula (18) also works for some non-compact metric graphs.

Example 2. *Consider the positive real line* $\mathbb{R}^+$ *again. Following our principle of not allowing bounces at vertices of degree 2, we can see that the only viable paths between two points* $x, y \in \mathbb{R}^+$ *is the direct path from* x *to* y *and the path from* x *to* 0 *to* y*, both with coefficient of 1. Therefore, we know that* $\mathcal{P}(x, y) = \{(x, y), (x, 0, y)\}$*. Additionally, we know that* $\alpha((x, y)) = \alpha((x, 0, y)) = 1$*, and* $\ell((x, y)) = |x - y|$ *and* $\ell((x, 0, y)) = |x + y|$*. Therefore, plugging these into the formula, we recover our original formula for the heat kernel:*

$$H(t, x, y) = \frac{1}{\sqrt{4\pi t}} \left(e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t} \right). \tag{25}$$

Note the small difference with the original formula. This is because we are working with different k *coefficients for the heat equation* $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$*.*

6. Infinite Metric Trees

6.1. **Definitions.** The heat kernel on (infinite) metric trees is not as well understood as compact metric graphs, and it is very difficult to compute an explicit value for the heat kernel on metric trees. This section will outline various methods to estimate the heat kernel on metric graphs.

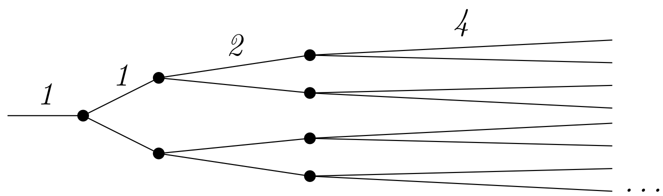
Note that we are assuming all metric trees to be of infinite volume, i.e. it is not bounded.

Definition 6.1. *For a metric tree* Γ*, the branching function is defined to be*

$$g_0(r) = \#\{x \in \Gamma : |x| = r\},$$

where $|x|$ *denotes the distance from* x *to the root.*

Example 3. *Let* Γ *be a binary tree with doubling edge lengths:*



Then, we have $g_0(r) = \min\{2^n : n \in \mathbb{N}, r < 2^n\}$*.*

6.2. **Results on Metric Trees.** The following theorem does not only apply to metric trees, but general metric graphs of infinite volume: [FK13, Theorem 2.2]

Theorem 6. *Let* Γ *be a connected graph of infinite volume. Then, for all* $x \in \Gamma$ *and* $t > 0$*,*

$$H(t, x, x) \leq (\pi t)^{-1/2}.$$

This is the sharpest possible bound, as evidenced in the following example:

Example 4. $\Gamma = \mathbb{R}_+$.

$$H(t, x, x) = \frac{1}{\sqrt{4\pi t}}(1 + e^{-\frac{x^2}{t}}) \leq \frac{1}{\sqrt{4\pi t}}(2) = \frac{1}{\sqrt{\pi t}}.$$

The long-time decay of the diagonal of the heat kernel on a metric tree is dependent on the speed of the growth of g_0.

Theorem 7. [FK13, Theorem 2.3] *If there exists a constant C_0 such that $g_0(2r) \leq C_0 g_0(r)$ for all $r \in [0, \infty)$ for some symmetric tree Γ, then there exists $c > 0$ such that for all $x \in \Gamma$ and $t > 0$,*

$$\frac{1}{c\sqrt{t} g_0(|x| + \sqrt{t})} \leq H(t, x, x) \leq \frac{c g_0(|x|)}{\sqrt{t} g_0(|x| + \sqrt{t})}.$$

If the metric tree does not fulfill the above condition, we still have a theorem for the upper bound for the diagonal of the heat kernel:

Theorem 8. [FK13, Theorem 2.5] *Let Γ be a symmetric tree, and assume that*

$$S_\Gamma^{-1}(\delta) := \sup_{r>0} \left(\int_0^r g_0(s)\, ds \right)^{(\delta-2)/\delta} \left(\int_r^\infty \frac{1}{g_0(s)}\, ds \right) < \infty,$$

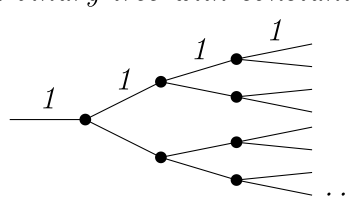
for some $\delta > 2$. Then, for all $x \in \Gamma$, $t > 0$,

$$H(t, x, x) \leq \left(\frac{\delta}{2\tilde{S}_\Gamma(\delta)} \right)^{\delta/2} t^{-\delta/2} g_0(|x|),$$

where

$$\tilde{S}_\gamma(\delta) := \left(\frac{(\delta-2)^{\delta-2} \delta^\delta}{(2(\delta-1))^{2(\delta-1)}} \right)^{1/\delta} S_\Gamma(\delta).$$

Remark. *The above theorem still imposes a condition on the speed of the growth of the tree. As a simple example, consider a binary tree with constant edge lengths of 1.*



For this tree, we can see that $g_0(r) = 2^{\lfloor r \rfloor}$, where $\lfloor r \rfloor$ denotes the largest integer that is smaller than or equal to r. We can quickly see that

$$\sup_{r>0} \left(\int_0^r g_0(s)\, ds \right)^{(\delta-2)/\delta} \left(\int_r^\infty \frac{1}{g_0(s)}\, ds \right) < \infty \tag{26}$$

if and only if $\frac{\delta-2}{\delta} = 1$, which is impossible. This can be attributed to the fact that the tree simply grows too quickly.

Definition 6.2. *We say that a graph Γ has global dimension $d \geq 1$ if*

$$0 < \inf_{r \geq 0} \frac{g_0(r)}{(1+r)^{d-1}} \leq \sup_{r \geq 0} \frac{g_0(r)}{(1+r)^{d-1}} < \infty. \tag{27}$$

Proposition 1. *If Γ has global dimension $d > 2$, then the condition in Theorem 8 is met, i.e.*

$$\sup_{r>0} \left(\int_0^r g_0(s)\, ds \right)^{(\delta-2)/\delta} \left(\int_r^\infty \frac{1}{g_0(s)}\, ds \right) < \infty,$$

with $\delta = d$.

Proof. Define $F(r) = \int_0^r g_0(s)\mathrm{d}s$, $G(r) = \int_r^\infty \frac{1}{g_0(s)}\mathrm{d}s$. We want to prove that

$$F(r)^{\frac{d-2}{d}} \cdot G(r) < \infty.$$

We know that for sufficiently large r, $g_0(r)$ grows about as fast as $(1+r)^{d-1}$. So, we can write

$$\begin{aligned} F(r) &= \int_0^r g_0(s)\mathrm{d}s \\ &\ll \int_0^r (1+r)^{d-1}\mathrm{d}s \\ &= \frac{(1+r)^d}{d}. \end{aligned}$$

On the other hand, we also write

$$\begin{aligned} G(r) &\ll \int_r^\infty \frac{1}{(1+s)^{d-1}}\mathrm{d}s \\ &= (1+r)^{2-d}. \end{aligned}$$

Putting it all together, we have

$$\begin{aligned} F(r)^{\frac{d-2}{d}} \cdot G(r) &\ll \left(\frac{(1+r)^d}{d}\right)^{\frac{d-2}{d}} \cdot (1+r)^{2-d} \\ &= \frac{(1+r)^{d-2}}{d^{\frac{d-2}{d}}} \cdot \frac{1}{(1+r)^{d-2}} \\ &= \frac{1}{d^{\frac{d-2}{d}}} < \infty. \end{aligned}$$

□

Proposition 2. *Suppose a graph Γ has global dimension $d \geq 1$. Then, Γ fulfills the volume doubling condition, or in other words, there exists C_0 such that:*

$$g_0(2r) \leq C_0 g_0(r), \tag{28}$$

for all $r \in [0, \infty)$.

Proof. Since Γ has global dimension d, this means that there exists some R such that for all $r > R$,

$$\Rightarrow \exists c_1, c_2 \in \mathbb{R} \text{ such that } c_1(1+r)^{d-1} \leq g_0(r) \leq c_2(1+r)^{d-1}.$$

Fix some large enough r_0 such that this holds. Then, we also get that

$$\begin{aligned} c_1(1+r_0)^{d-1} \leq g_0(2r_0) &\leq c_2(1+r_0)^{d-1} \\ \Rightarrow \frac{g_0(2r_0)}{g_0(r_0)} &\leq \frac{c_2(1+2r_0)^{d-1}}{c_1(1+r_0)^{d-1}}. \end{aligned}$$

Therefore, for large r, the ratio $\frac{g_0(2r)}{g_0(r)}$ is bounded by $k_1 \frac{c_2}{c_1}$ for some k_1.

Now, for small r, we can observe that we can pick $r_0 \in (0, r)$, and we can simply take $k_2 = \sup_{r_0 \in [0,r)} \frac{g_0(2r_0)}{g_0(r_0)}$. Since g_0 is an integer-valued function and r is a fixed finite value, $k_2 < \infty$.

Let $C_0 = \max(k_1, k_2)$ and we are done. □

Proposition 3. *The converse of Proposition 2 holds, i.e. if there exists* C_0 *such that* $g_0(2r) \leq C_0 g_0(r)$ *for all* $r \in [0,\infty)$*, then there exists* $d \geq 1$ *such that*

$$0 < \inf_{r\geq 0} \frac{g_0(r)}{(1+r)^{d-1}} \leq \sup_{r\geq 0} \frac{g_0(r)}{(1+r)^{d-1}} < \infty. \tag{29}$$

Proof. Suppose that $g_0(2r) \leq C_0 g_0(r)$ for all $r \in [0,\infty)$. Then, let $g_0(1) = k < \infty$. Using the volume doubling condition, we can see $g_0(2) \leq C_0 k$, $g_0(4) \leq C_0^2 k$, and so on. Linearly interpolating this, we can therefore see that $g_0(r) \leq C_0^{\log_2(r)}$ for all r.

Using this information, we can pick d such that $C_0^{\log_2(r)} \leq k(1+r)^{d-1}$, which we know exists since C_0 is a finite value. Therefore, Γ has global dimension d as required. □

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Department of Mathematics, University of Michigan, Ann Arbor, MI 48104 USA
Email address: justliu@umich.edu