CONFORMAL INVARIANCE OF CLIFFORD MONOGENIC FUNCTIONS

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Abstract. Clifford algebras generalize many features of complex numbers and quaternions. Quaternionic analogues of holomorphic functions – called regular functions – are known to be invariant under conformal or Möbius transformations. It is also known that a similar invariance property holds in the context of Clifford algebras associated to positive definite quadratic forms. In this project, we generalize these results to the case of Clifford algebras associated to non-degenerate quadratic forms. This approach puts the indefinite signature case on the same footing as the classical positive definite case.

1. Introduction

It is known that a class of functions that generalizes holomorphic functions in complex analysis, called regular functions in the context of quaternions and monogenic functions in the context of Clifford algebras, is preserved under a certain multiplier representation of conformal transformations. See [11] for the quaternion case and [9] for Clifford algebras of Euclidean space with positive definite signature.

Let $\mathrm{Cl}(V)$ be the universal Clifford algebra associated to a vector space V with quadratic form Q, and let $f : V \to \mathrm{Cl}(V)$ be a monogenic function. If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ represents the conformal (or Möbius) transformation on V

$$x \mapsto (ax+b)(cx+d)^{-1}, \tag{1}$$

then it is known in the case of positive definite quadratic forms Q that

$$J_A(x)f(Ax) = \frac{(cx+d)^{-1}}{|(x\tilde{c}+\tilde{d})(cx+d)|^{n/2-1}} f(Ax) \tag{2}$$

is also monogenic. In this paper, we extend this result to the case of quadratic forms Q having mixed signature. While the formula in the mixed signature case visually appears the same, its meaning is slightly different. First of all, the transformation (1) takes place on the conformal closure of the vector space V, and it is different from the one-point compactification of V. Secondly, one needs to revisit the definition of the monogenic functions in this context and choose the "right" Dirac operator. We argue that there is a certain natural choice of the Dirac operator that makes the result valid, while other choices would not work.

We begin by reviewing the ambient construction of conformal compactification and conformal transformations, following [10]. After we introduce Clifford algebras based on [4, 5], we devote a considerable part of the paper to unifying the ambient construction with the Clifford algebra generalization of the description of Möbius transformation via 2×2 matrices. In this generalization, conformal transformations are expressed via Vahlen matrices with Clifford algebra valued entries. To unify the two treatments of conformal transformation, we build upon [1, 7] and provide details that might be left out.

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Finally, we verify that the multiplier presentation of Vahlen matrices acting on monogenic functions described in eq. (2) remains valid for vector spaces with non-degenerate signature. Such presentation frequently appears in the context of Clifford analysis such as in [9], but our approach is a more geometric and is independent of the signature of the quadratic form Q.

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2. Conformal Transformations

In this section, we review the definition of the conformal compactification N of a vector space V and the conformal action of $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$ on $N(V)$. For detailed exposition, readers may refer to [10].

2.1. **Basic Definitions of Conformal Transformations.**

Let p, q be nonnegative integers with $n = p + q \geq 2$, and let V be an n-dimensional vector space equipped with a symmetric non-degenerate bilinear form $g : V \times V \to \mathbb{R}$ with signature (p, q). Having signature (p, q) means that there exists a linear bilinear form preserving isomorphism between $\mathbb{R}^{p,q}$ with V where $\mathbb{R}^{p,q}$ is the n-dimensional vector space $\mathbb{R}^n$ with bilinear form

$$g(x, y) = x^1y^1 + \cdots + x^py^p - x^{p+1}y^{q+1} - \cdots - x^{p+q}y^{q+q}. \tag{3}$$

Definition 1. Let M be a manifold equipped with a symmetric bilinear form $g : TM \times TM \to \mathbb{R}$. We say that another symmetric bilinear form $\hat{g} : TM \times TM \to \mathbb{R}$ is conformally equivalent to g if there exists a smooth function $\Omega : M \to \mathbb{R}_{>0}$ such that $\hat{g} = \Omega^2 g$.

A smooth map $\phi : (M, g) \to (M', g')$ between manifolds equipped with a symmetric bilinear forms is conformal if the pullback of g' along ϕ is conformally equivalent to g. That is, there exists a smooth function $\Omega : M \to \mathbb{R}_{>0}$, called the conformal factor, such that

$$\phi^* g' = \Omega^2 g. \tag{4}$$

A conformal transformation on a manifold M is a conformal map $\phi : U \to M$, where $U \subseteq M$ is a non-empty connected open subset.

If we consider bilinear forms that are related by positive scalar fields as equivalent, then conformal maps are exactly the smooth maps that preserve this equivalence class of bilinear forms. We call this equivalence class of bilinear forms a conformal structure. Note that our definition of conformal transformation is not necessarily orientation preserving. To prevent terminology conflicts, in section 2.3, we avoid talking about the conformal group, which, by definition, is the connected component of the group of all orientation preserving conformal transformation. Instead, we speak of the group of conformal transformations in the sense of definition 1.

2.2. **Conformal Compactification.**

In this section, we describe the ambient construction of conformal compactification. The ambient construction of conformal compactification of a vector space V with symmetric non-degenerate bilinear form g with signature (p, q) takes place in $V \oplus \mathbb{R}^{1,1}$. By abuse of notation, we denote the bilinear form on $V \oplus \mathbb{R}^{1,1}$ by g and let (e_-, e_+) be the standard basis of $\mathbb{R}^{1,1}$ with

$$g(e_+, e_+) = 1, \quad g(e_-, e_-) = -1, \quad g(e_+, e_-) = g(e_-, e_+) = 0. \tag{5}$$

Lemma 2. *Consider the line $\{(t,s)|t+s=1\} \subseteq \mathbb{R}^{1,1}$, any section $r : M \to M \times \{(t,s)|t+s = 1\} \subseteq M \times \mathbb{R}^{1,1}$ is a local isometry.*

Proof. Any section r is of the form

$$r(x) = x + t(x)e_- + (1 - t(x))e_+. \tag{6}$$

for some function $t : M \to \mathbb{R}$. The pushforward is then

$$r_*(v) = v + (\partial_v t)e_- - (\partial_v t)e_+. \tag{7}$$

It is then clear that for all v, u

$$\begin{aligned} r^* g(v, u) &= g(v + (\partial_v t)e_- - (\partial_v t)e_+, u + (\partial_u t)e_- - (\partial_u t)e_+) \\ &= g(v, u). \end{aligned} \tag{8}$$

Since $r^* g = g$, r is a local isometry. □

Let

$$\mathcal{N} = \{z \in V \oplus \mathbb{R}^{1,1} \mid z \neq 0,\ g(z, z) = 0\}. \tag{9}$$

be the null cone in $V \oplus \mathbb{R}^{1,1}$. Then $\mathcal{N}$ inherits metric from $V \oplus \mathbb{R}^{1,1}$. Let $i : V \to \mathcal{N}$ be defined by

$$i(x) = x + \frac{1 + g(x, x)}{2} e_- + \frac{1 - g(x, x)}{2} e_+. \tag{10}$$

The map i is constructed so that lemma 2 applies and i is a local isometry.

Lemma 3. *Suppose $x \in \mathcal{N}$, then we have $g(x, v) = 0$ for all tangent vector $v \in T_x\mathcal{N}$.*

Proof. Let $s : (-\epsilon, \epsilon) \to \mathcal{N}$ ba a path that represents v, that is $s(0) = x$ and $\frac{ds}{dt}(0) = v$. We have

$$g(x, v) = \left.\frac{d}{dt}\right|_{t=0} g(x, s(t)) = \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} g(s(t), s(t)) = 0 \tag{11}$$

because $g(s(t), s(t)) = 0$ for all t. □

Lemma 4. *For arbitrary smooth map $\phi : V \to \mathcal{N}$ and smooth function $\Omega : V \to \mathbb{R}_{>0}$, the function $(\Omega\phi)(x) = \Omega(x)\phi(x)$ satisfies*

$$(\Omega\phi)^* g = \Omega^2 \phi^* g. \tag{12}$$

Proof. By product rule, we have

$$(\Omega\phi)_*(u) = \Omega\phi_*(u) + (\partial_u \Omega)\phi. \tag{13}$$

By lemma 3, we have $g(\phi, v) = 0$ for all v tangent to the null cone. Therefore, we have

$$\begin{aligned} (\Omega\phi)^* g(u, v) &= g(\Omega\phi_*(u) + (\partial_u\Omega)\phi, \Omega\phi_*(v) + (\partial_v\Omega)\phi) \\ &= g(\Omega\phi_*(u), \Omega\phi_*(v)) \\ &= \Omega^2 \phi^* g(u, v). \end{aligned} \tag{14}$$

□

Based on the preceding lemma, we observe that $\Omega i : V \to \mathcal{N}$ is a conformal map with conformal factor Ω. The core idea of conformal compactification involves selecting a slice S of $\mathcal{N}$ such that the canonical projection $\pi : \mathcal{N} \to N = \mathcal{N}/\mathbb{R}^\times$ is a local diffeomorphism when restricted to S. Through the local diffeomorphism, we can endow N with a symmetric bilinear form such that π is an isometry, thereby giving N a conformal structure. Subsequently, we can determine Ω such that $\Omega i : V \to S$ has image in S, then $\pi \circ (\Omega i) : V \to N$ will become a conformal embedding.

In the positive definite case, we can simply choose the slice to be the intersection between $\mathcal{N}$ and $V \times \{(t, s) \mid t = 1\}$. This choice leads to us to

$$\Omega(x) = \frac{2}{1 + g(x, x)}, \tag{15}$$

so that

$$\Omega i(x) = \frac{2x}{1 + g(x, x)} + e_- + \frac{1 - g(x, x)}{1 + g(x, x)} e_+, \tag{16}$$

which is exactly the stereographic projection up to sign convention.

In the indefinite case, the situation becomes more complicated because $g(x, x)$ can be -1, rendering eq. (15) undefined. To tackle this, we follow the standard approach described in [10] for conformal compactification of $\mathbb{R}^{p,q}$. The approach can be applied to conformal compactification of V upon choosing a basis. While choosing a basis is not a canonical procedure, there might be an alternative canonical construction.

Let the slice S be the intersection between $\mathcal{N}$ and the hypersphere

$$\mathbb{S}^{n+2} = \{(z_0, \dots, z_{n+1}) \in \mathbb{R}^{p+1,q+1} \mid z_0^2 + \cdots + z_{n+1}^2 = 2\}, \tag{17}$$

thus yielding

$$S = \left\{ z \in \mathbb{R}^{p+1,q+1} \;\middle|\; \sum_{j=0}^{p} (z^j)^2 = 1 = \sum_{j=p+1}^{n+1} (z^j)^2 \right\} \cong \mathbb{S}^p \times \mathbb{S}^q. \tag{18}$$

Then, it can be shown that $\pi : S \to N$ is a local diffeomorphism and a double covering. Moreover, the metric g of S induced by inclusion $S \subseteq \mathbb{R}^{p+1,q+1}$ carries over to N such that π becomes a local isometry. One can show that

$$\Omega(x) = \frac{2}{\sqrt{1 + 2\sum_{j=1}^{n} (x^j)^2 + g(x, x)^2}} \tag{19}$$

indeed gives rise to $\Omega i : V \to S$ and thus $\pi \circ (\Omega i) : V \to N$ is a conformal embedding.

2.3. **The Group of Conformal Transformations.** Let us review some well known results concerning the group of conformal transformation. Reader may again refer to [10] for details, bearing in mind that our notion of group of conformal transformations includes non-orientation preserving transformations.

Proposition 5. *For $p + q > 2$, every conformal transformation $\phi : U \to \mathbb{R}^{p,q}$ is a finite composition of the following types of conformal transformations:*

1. *(Translation) $x \mapsto x + c$ where $c \in V$;*
2. *(Orthogonal transformation) $x \mapsto Ax$ where $A \in \mathrm{O}(V)$;*
3. *(Dilation) $x \mapsto \lambda x$ where $\lambda > 0$;*
4. *(Inversion) $x \mapsto \frac{x}{g(x,x)}$.*

Henceforth, our discussion will focus exclusively on the case $p+q>2$.

Let us denote the conformal compactification of V by $N(V)$, and the canonical conformal embedding by $i: V \to N(V)$. The main result is as follows.

Proposition 6. *For $p+q>2$, every conformal transformation $\phi: U \to V$ can be uniquely extended to a conformal diffeomorphism on $N(V)$, which is a diffeomorphism $\hat{\phi}: N(V) \to N(V)$ satisfying $\hat{\phi} \circ i = i \circ \phi$.*

Moreover, every conformal diffeomorphism $N(V) \to N(V)$ can be induced by an action of $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$. The group of all conformal diffeomorphisms $N(V) \to N(V)$ is isomorphic to $\mathrm{O}(V \oplus \mathbb{R}^{1,1})/\{\pm 1\}$.

The proof of this proposition involves applying proposition 5 and expressing the four types of conformal transformations as actions of elements of $\mathrm{O}(V \oplus \mathbb{R}^{1,1})$. Let us provide the expressions to better understand conformal transformations. For clarity and geometric intuition, we find it most advantageous to work with a new basis $n_0 = (e_- + e_+)/2$ and $n_\infty = (e_- - e_+)/2$ for $\mathbb{R}^{1,1}$. Note that $n_0 = i(0)$ represents the zero vector in V, and n_∞ represents a point at infinity (the inversion of the zero vector). In this basis, the symmetric bilinear form is given by

$$g(n_0, n_0) = g(n_\infty, n_\infty) = 0, \quad g(n_0, n_\infty) = g(n_\infty, n_0) = -\frac{1}{2}, \tag{20}$$

and eq. (10) is given by

$$i(x) = x + n_0 + g(x,x)n_\infty. \tag{21}$$

(1) Translation. For a translation $x \mapsto x + c$, we have the orthogonal transformation

$$x \mapsto x + 2g(x,c)n_\infty, \quad n_0 \mapsto c + n_0 + g(c,c)n_\infty, \quad n_\infty \mapsto n_\infty. \tag{22}$$

From calculation, we see that

$$\begin{aligned} i(x) &\mapsto (x + 2g(x,c)n_\infty) + (c + n_0 + g(c,c)n_\infty) + g(x,x)n_\infty \\ &= (x+c) + n_0 + g(x+c, x+c)n_\infty \\ &= i(x+c). \end{aligned} \tag{23}$$

(2) Orthogonal transformation. For an orthogonal transformation $x \mapsto Ax$, we simply have

$$x \mapsto Ax, \quad n_0 \mapsto n_0, \quad n_\infty \mapsto n_\infty. \tag{24}$$

It is clear that $i(x) \mapsto i(Ax)$.

(3) Dilation. For a dilation $x \mapsto rx$, we have

$$x \mapsto x, \quad n_0 \mapsto \frac{1}{r}n_0, \quad n_\infty \mapsto r n_\infty. \tag{25}$$

Using the fact that proportional vectors in $\mathcal{N}$ are identified, we see that

$$\begin{aligned} i(x) &\mapsto x + \frac{1}{r}n_0 + rg(x,x)n_\infty \\ &\sim rx + n_0 + g(rx, rx)n_\infty \\ &= i(rx). \end{aligned} \tag{26}$$

(4) Inversion. For inversion $x \mapsto \frac{x}{g(x,x)}$, we have

$$x \mapsto x, \quad n_0 \mapsto n_\infty, \quad n_\infty \mapsto n_0, \tag{27}$$

or equivalently $e_- \mapsto e_-$ and $e_+ \mapsto -e_+$. We can see that for x that is not null (i.e. $g(v,v) \neq 0$, also known as anisotropic),

$$\begin{aligned} i(x) &\mapsto x + g(x,x)n_0 + n_\infty \\ &= \frac{x}{g(x,x)} + n_0 + \frac{1}{g(x,x)}n_\infty \\ &= i\left(\frac{x}{g(x,x)}\right). \end{aligned} \tag{28}$$

2.4. **Classification of Points in $N(V)$.** The points in $N(V)$ are represented by $a + be_- + ce_+ \neq 0$ such that $g(a,a) - b^2 + c^2 = 0$, where $a \in V$, $b, c \in \mathbb{R}$. Proportional elements are identified. We can distinguish between the following three classes.

(1) $b = \frac{1+g(a,a)}{2}$ and $c = \frac{1-g(a,a)}{2}$. These points are identified with vectors in V through eq. (10). This class is closed under translation, orthogonal transformation, and dilation. The subset of points that represents vectors in V that are not null is closed under inversion.
(2) $g(a,a) = 0$, $b = -\frac{1}{2}$ and $c = \frac{1}{2}$. Such a point represents the inversion of null vectors $a \in V$.
(3) $g(a,a) = b = c = 0$ and $a \neq 0$. We can think of these points as the limiting points of the null lines generated by the null vectors $a \in V$.

3. Clifford Algebras

In this section, we review the fundamentals of Clifford algebras including its associated groups and introduce monogenic functions.

3.1. **Definitions and Constructions of Clifford Algebras.** Conventions vary greatly when in comes to Clifford algebras. In this paper, we closely adhere to the convention found in Garling [5]. Let V be a vector space over a field K with quadratic form $Q : V \to K$.

Definition 7. A Clifford algebra W of V is a unital associative algebra W that is equipped with an injective linear map $j : V \to W$ such that

(1) $j(x)^2 = -Q(x)$ for all $x \in V$,
(2) $1 \notin j(V)$, and
(3) W is generated by K and $j(V)$.

Some authors such as Delanghe, Sommen, and Souček [4] use the sign convention where $j(x)^2 = Q(x)$. Our sign convention aligns with the prevailing practice in Clifford analysis and in the related literature concerning our main inquiry in the positive definite case.

Suppose W is a Clifford algebra of V. When there is no ambiguity, we identify V as a subspace of W through the defining map $j : V \to W$. For example, we can say that for $x \in V$, we have $x^2 = -Q(x)$.

Several equivalent approaches exist for defining the universal Clifford algebra associated to V. We employ the universal property described in [5].

Definition 8. The universal Clifford algebra $\mathrm{Cl}(V)$ of V is a Clifford algebra with the universal property that for each isometry $\phi : V \to V'$ and each Clifford algebra W' of V', there exists a unique algebra homomorphism $\hat{\phi} : \mathrm{Cl}(V) \to W'$ such that the following diagram commutes.

$$\begin{array}{ccc} V & \xrightarrow{\phi} & V' \\ \big\downarrow & & \big\downarrow \\ \mathrm{Cl}(V) & \xrightarrow{\hat{\phi}} & W' \end{array} \tag{29}$$

We can construct the universal Clifford algebra explicitly as a quotient of the tensor algebra $\bigotimes V = \bigoplus_{k=0}^{\infty} V^{\otimes k}$. This construction is due to Chevalley [2], which we now describe. Let I be the two sided ideal in $\bigotimes V$ generated by the elements of the form

$$x \otimes x + Q(x) \quad \text{for } x \in V \tag{30}$$

Then the quotient algebra

$$\mathrm{Cl}(V) = \bigotimes V / I \tag{31}$$

is the universal Clifford algebra of V.

The universal property guarantees the uniqueness of the universal Clifford algebra of V up to isomorphism by the standard argument. The explicit construction serves to establish the existence of such a universal object. It can be easily checked that the quotient algebra is a Clifford algebra of V. The universal property in definition 8 can also be readily deduced using the universal properties of tensor algebra and quotient. This realization leads to the following universal property.

Proposition 9. *The universal Clifford algebra* $\mathrm{Cl}(V)$ *of* V *with* $i : V \to \mathrm{Cl}(V)$ *has the universal property that for every unital associative algebra* W *with a linear map* $h : V \to W$ *such that* $h(x)^2 = -Q(x)$ *for all* $x \in V$*, there exists a unique algebra homomorphism* $\hat{h} : \mathrm{Cl}(V) \to W$ *such that* $h = \hat{h} \circ i$.

3.2. **Basis of a Clifford Algebra.** For a real vector space V, there is a canonical bijection between quadratic forms on V and bilinear forms on V. We would like to establish this connection to talk about Clifford algebra of a vector space equipped with a symmetric bilinear form.

Proposition 10. *Let* V *be a vector space over a field* K *that is not of characteristic* 2*. For each quadratic form* Q *on* V*, there corresponds a symmetric bilinear form* g *given by*

$$g(x, y) = \frac{1}{2}[Q(x + y) - Q(x) - Q(y)], \tag{32}$$

and, conversely, for each symmetric bilinear form g*, there corresponds a quadratic form given by*

$$Q(x) = g(x, x). \tag{33}$$

Moreover, this correspondence is a bijection.

Proof. Given quadratic form Q, the quadratic form $\hat{Q}$ that corresponds to the symmetric bilinear form corresponding to Q is

$$\begin{aligned}\hat{Q}(x) &= \frac{1}{2}[Q(x+x) - Q(x) - Q(y)] \\ &= \frac{1}{2}[4Q(x) - Q(x) - Q(x)] \\ &= Q(x),\end{aligned} \tag{34}$$

which is the same as Q. Given symmetric bilinear form g, the symmetric bilinear form $\hat{g}$ that corresponds to the quadratic form corresponding to g is

$$\begin{aligned}\hat{g}(x,y) &= \frac{1}{2}[g(x+y,x+y) - g(x,x) - g(y,y)] \\ &= \frac{1}{2}[g(x,y) + g(y,x)] \\ &= g(x,y),\end{aligned} \tag{35}$$

which is the same as g. □

We denote by $\mathbb{R}^{p,q}$ the generalized Minkowski space. This is the vector space $\mathbb{R}^{p+q}$ with the bilinear form eq. (3). Now, we know that it is also equipped with a quadratic form, and we can consider its Clifford algebra $\mathrm{Cl}(\mathbb{R}^{p,q})$.

Proposition 11. *Let V be a real vector space with a symmetric bilinear form g. If $x, y \in V \subseteq \mathrm{Cl}(V)$, then $xy + yx = -2g(x, y)$.*

Proof. Direct calculation using eq. (32) shows

$$xy + yx = (x+y)^2 - x^2 - y^2 = -Q(x+y) + Q(x) + Q(y) = -2g(x,y), \tag{36}$$

where Q is the quadratic form corresponding to g. □

Corollary 12. *If $x, y \in V$ are orthogonal (i.e. $g(x, y) = 0$), then $xy = -yx$.*

Theorem 13. *The Clifford algebra $\mathrm{Cl}(\mathbb{R}^{p,q})$ has dimension 2^n as a vector space, where $n = p + q$. Specifically, it has a basis consists of elements of the form $e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$ where $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n$, and where (e_i) is the standard basis of $\mathbb{R}^{p+q}$.*

One can speculate that this is true based on the corollary 12. For a concrete proof, see [4] for the identification between Clifford algebra and exterior algebra with a new product operation or see [5] for the identification between Clifford algebra and certain linear transformations on the exterior algebra.

For convenience, let $N = \{1, 2, \ldots, n\}$ and let $e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$ where $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le n$ be denoted by e_A if $A = \{\alpha_1, \ldots, \alpha_k\}$. We call $\{e_A \mid A \subseteq N\}$ the standard basis for the Clifford algebra $\mathrm{Cl}(\mathbb{R}^{p,q})$.

There are three important involutions defined on the Clifford algebra.

Definition 14. Let $i : V \to \mathrm{Cl}(V)$ be the canonical embedding into Clifford algebra.

- The grade involution $\hat{\cdot} : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is induced by $h : V \to \mathrm{Cl}(V)$ defined by $h(x) = -x$. For vectors $\{v_1, v_2, \ldots, v_k\} \subseteq V$, we have

$$v_1 \widehat{v_2 \cdots} v_k = (-1)^k v_1 v_2 \cdots v_k. \tag{37}$$

Thus, in the standard basis, the grade involution is given by

$$\widehat{e_A} \mapsto (-1)^{|A|} e_A \tag{38}$$

where $|A|$ is the cardinality of A.

- The reversal $\tilde{\cdot} : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is the transpose operation on the level of tensor algebra which descends to Clifford algebra because the two sided ideal I in definition 8 is preserved. The transpose operation is defined on each summand $V^{\otimes k}$ of tensor algebra as

$$\begin{aligned} (\cdot)^{\mathsf{t}} : V^{\otimes k} &\to V^{\otimes k} \\ v_1 \otimes \cdots \otimes v_k &\mapsto v_k \otimes \cdots \otimes v_1. \end{aligned} \tag{39}$$

Hence, for vectors $\{v_1, \ldots, v_k\} \subseteq V$, we have

$$\widetilde{v_1 \cdots v_k} = v_k \cdots v_1. \tag{40}$$

In the standard basis, the reversal is given by

$$\widetilde{e_A} = (-1)^{\frac{|A|(|A|-1)}{2}} e_A. \tag{41}$$

Note that the reversal is not an algebra homomorphism, but an anti-homomorphism.

- The Clifford conjugation is defined as a composition of the grade involution and the reversal (note that these two involutions commute). We denote the Clifford-conjugation by $\bar{a} = \hat{\tilde{a}}$. In the standard basis, the Clifford conjugation is given by

$$\overline{e_A} = (-1)^{\frac{|A|(|A|+1)}{2}} e_A. \tag{42}$$

Like the reversal, the Clifford conjugation is an algebra anti-homomorphism.

One important observation is that the grade involution, the reversal, and the Clifford conjugation all commute with taking the multiplicative inverse in $\mathrm{Cl}(V)$. For example,

$$\hat{a}\widehat{(a^{-1})} = \widehat{aa^{-1}} = 1 = \widehat{a^{-1}a} = \widehat{(a^{-1})}\hat{a} \tag{43}$$

shows that $\widehat{(a^{-1})} = \hat{a}^{-1}$. We will use this property implicitly.

3.3. **Associated Groups.** An essential feature of Clifford algebra is that we can perform reflections using the so-called twisted adjoint action. Let V be a vector space with symmetric bilinear form g. Let $v \in V$ be a vector that is not null, hence every vector $x \in V$ splits into $x = x^{\perp} + \lambda v$ with $x^{\perp} \in V$ orthogonal to v. The explicit decomposition is given by

$$x^{\perp} = x - \frac{g(x,v)}{g(v,v)} v \quad \text{and} \quad \lambda = \frac{g(x,v)}{g(v,v)}. \tag{44}$$

Then, the map $\sigma_v : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ defined as

$$\sigma_v(x) = -vxv^{-1} \tag{45}$$

is a reflection on the space V in the direction of v because

$$\begin{aligned} \sigma_v(x^{\perp} + \lambda v) &= -vx^{\perp}v^{-1} - v(\lambda v)v^{-1} \\ &= x^{\perp} - \lambda v. \end{aligned} \tag{46}$$

Theorem 15 (Cartan-Dieudonné)**.** *Let V be a vector space with non-degenerate symmetric bilinear form, then every orthogonal linear transformation on V can be expressed as the product of at most* $\dim V$ *reflections in the direction of vectors that are not null.*

Thus, by the Cartan-Dieudonné theorem (see [5] for its proof), all orthogonal linear transformations on V can be expressed using a twisted adjoint action

$$\sigma_a : x \mapsto ax\hat{a}^{-1} \tag{47}$$

where a is a product of vectors that are not null.

In a sense, the twisted adjoint action specializes in orthogonal transformations. To make this precise, let us consider all elements in Clifford algebra such that their twisted adjoint action preserves the vector space.

Definition 16. The Lipschitz group $\Gamma(V)$ consists of elements in $\mathrm{Cl}(V)$ that preserve vector space V under the twisted adjoint action, that is

$$\Gamma(V) = \{a \in \mathrm{Cl}^\times(V) \mid ax\hat{a}^{-1} \in V \text{ for all } x \in V\} \tag{48}$$

where $\mathrm{Cl}^\times(V)$ is the set of invertible elements in $\mathrm{Cl}(V)$.

Lemma 17. *For all $a \in \Gamma(V)$, σ_a is always an orthogonal transformation. Moreover, $\sigma : \Gamma(V) \to \mathrm{O}(V)$ is surjective.*

Proof. For all $x \in V$, we have

$$\begin{aligned} Q(\sigma_a(x)) &= (ax\hat{a}^{-1})(\widehat{ax\hat{a}^{-1}}) \\ &= -(ax\hat{a}^{-1})(\hat{a}xa^{-1}) \\ &= Q(x). \end{aligned} \tag{49}$$

Surjectivity follows directly from the Cartan-Dieudonné theorem. □

Lemma 18. *The kernel of $\sigma : \Gamma(V) \to \mathrm{O}(V)$ is $\mathbb{R}^\times$. Equivalently, we have the following exact sequence*

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(V) \xrightarrow{\sigma} \mathrm{O}(V) \longrightarrow 1. \tag{50}$$

Proof. It is clear that $\mathbb{R}^\times \subseteq \ker\sigma$, so it remains to prove that $\ker\sigma \subseteq \mathbb{R}^\times$. Let (e_i) be an orthonormal basis for V and $\{e_A \mid A \subseteq N\}$ be the corresponding basis for $\mathrm{Cl}(V)$. Let us suppose $a \in \ker\alpha$ and write it as $a = \sum_{A\subseteq N} \lambda_A e_A$. Since $ae_i\hat{a}^{-1} = e_i$ for all i, we have

$$\sum_{A\subseteq N} (-1)^{|A|}\lambda_A e_i e_A = e_i\hat{a} = ae_i = \sum_{A\subseteq N} \lambda_A e_A e_i. \tag{51}$$

Therefore, we obtain

$$(-1)^{|A|}\lambda_A e_i e_A = \lambda_A e_A e_i \tag{52}$$

for all i and A, but this expression only holds either when $\lambda_A = 0$ or when $i \notin A$. The fact that it holds for all i implies that $\lambda_A = 0$ for all nonempty A. Hence, $a = \lambda_\emptyset \in \mathbb{R}^\times$. □

Theorem 19. *Let V be n-dimensional. The Lipschitz group is precisely the group generated by $\mathbb{R}^\times$ and the invertible vectors (note that a vector is invertible if and only if it is not null). Moreover, every element in the Lipschitz group $\Gamma(V)$ can be expressed as a product of $\mathbb{R}^\times$ and at most n invertible vectors.*

Proof. For every $a \in \Gamma(V)$, there exists a product of at most n invertible vectors $v_1 \cdots v_k$ such that $\sigma_a = \sigma_{v_1\cdots v_k}$ by the Cartan-Dieudonné theorem. By the previous lemma, we have $a = \lambda v_1 \cdots v_k$ for some $\lambda \in \ker\sigma = \mathbb{R}^\times$. □

Corollary 20. *If $a \in \mathrm{Cl}(V)$, then $a \in \Gamma(V)$ if and only if $a\bar{a} \in \mathbb{R}^\times$ and $ax\tilde{a} \in V$ for all $x \in V$.*

Proof. Suppose $a \in \Gamma(V)$, then we can write $a = v_1 v_2 \cdots v_k$ as a product of vectors $v_1, \dots, v_k$ that are not null. It is clear that $a\bar{a} = Q(v_1)\cdots Q(v_k) \in \mathbb{R}^\times$. Fix arbitrary $x \in V$, since $\bar{a} = a^{-1}(a\bar{a})$, we have

$$ax\tilde{a} = ax\hat{\bar{a}} = ax\hat{a}^{-1}(a\bar{a}) \in V. \tag{53}$$

Conversely, if $a\bar{a} \in \mathbb{R}^\times$, then $a \in \mathrm{Cl}^\times(V)$. And $ax\tilde{a} \in V$ for all $x \in V$ implies that

$$ax\hat{a}^{-1} = \hat{a}xa^{-1} = -\frac{\overline{ax\tilde{a}}}{a\bar{a}} \in V, \tag{54}$$

for all $x \in V$. Thus, $a \in \Gamma(V)$. □

Corollary 21. *If $a \in \Gamma(V)$, then $\bar{a}xa \in V$ for all $x \in V$.*

Proof. Write $a = v_1 v_2 \cdots v_k$ as a product of vectors $v_1, \dots, v_k$ that are not null, then we have

$$\begin{aligned} \bar{a}xa &= (-1)^k v_k \cdots v_1 x v_1 \cdots v_k \\ &= (-1)^k v_k \cdots v_1 x \widetilde{v_k \cdots v_1}, \end{aligned} \tag{55}$$

which is in V whenever $x \in V$ because $v_k \cdots v_1 \in \Gamma(V)$. □

Definition 22. The pin group $\mathrm{Pin}(V)$ and the spin group $\mathrm{Spin}(V)$ are defined as

$$\begin{aligned} \mathrm{Pin}(V) &= \{a \in \Gamma(V) \mid \bar{a}a = \pm 1\}, \\ \mathrm{Spin}(V) &= \{a \in \Gamma(V) \mid \bar{a}a = \pm 1,\ \hat{a} = a\}. \end{aligned} \tag{56}$$

Note that the spin group induces orientation preserving transformations on V (even number of reflections) through the twisted adjoint action. Furthermore, the pin group and the spin group are double covers of the orthogonal group and special orthogonal group respectively. This can be summarized by the following exact sequences

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\sigma} \mathrm{O}(V) \longrightarrow 1, \tag{57}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\sigma} \mathrm{SO}(V) \longrightarrow 1. \tag{58}$$

Relating back to the conformal transformations, observe that the volume form ω (i.e. the product of a set of orthonormal basis vectors) acts as the total reflection $v \mapsto -v$, under the twisted adjoint action Thus, proposition 6 can be rephrased as the following.

Proposition 23. *When $\dim V > 2$, the group of all conformal diffeomorphisms $N(V) \to N(V)$ is isomorphic to $\mathrm{Pin}(V \oplus \mathbb{R}^{1,1})/\{\pm 1, \pm\omega\}$.*

3.4. **Monogenic Functions.** Consider $\mathbb{R}^{p,q}$ and a left $\mathrm{Cl}(\mathbb{R}^{p,q})$-module M. The Dirac operator D with respect to the standard basis is defined to act on real differentiable functions $f : \mathbb{R}^{p,q} \to M$ by

$$\begin{aligned} Df &= e_1 \frac{\partial f}{\partial x^1} + \cdots + e_p \frac{\partial f}{\partial x^p} - e_{p+1}\frac{\partial f}{\partial x^{p+1}} - \cdots - e_{p+q}\frac{\partial f}{\partial x^{p+q}} \\ &= \begin{pmatrix} e_1 & \cdots & e_{p+q} \end{pmatrix} \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix} \begin{pmatrix} \partial/\partial x^1 \\ \vdots \\ \partial/\partial x^{p+q} \end{pmatrix} f \end{aligned} \tag{59}$$

Note that the Dirac operator is independent of the orthonormal basis chosen because if

$$\begin{pmatrix} e'_1 \\ \vdots \\ e'_2 \end{pmatrix} = T \begin{pmatrix} e_1 \\ \vdots \\ e_2 \end{pmatrix} \tag{60}$$

where $T \in \mathrm{O}(p,q)$ by definition satisfies

$$T^{\intercal} \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix} T = \begin{pmatrix} 1_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & -1_{q\times q} \end{pmatrix}, \tag{61}$$

then the partial derivatives also transform by

$$\begin{pmatrix} \partial/\partial x'^1 \\ \vdots \\ \partial/\partial x'^{p+q} \end{pmatrix} = T \begin{pmatrix} \partial/\partial x^1 \\ \vdots \\ \partial/\partial x^{p+q} \end{pmatrix}. \tag{62}$$

Substitution into eq. (59) reveals that the Dirac operator is independent of the choice of an orthonormal basis. We can do even better and define Dirac operator in a basis independent fashion.

Definition 24. For vector space V with non-degenerate symmetric bilinear form g, which we think of it as a map $V \to V^*$ or as an element of $V^* \otimes V^*$. Non-degeneracy implies that $g : V \to V^*$ is an isomorphism, so g can also be considered an element in $V \otimes V$.

Let L be the space of linear operators on functions $V \to M$ where M is a left $\mathrm{Cl}(V)$-module. Define the map $i : V \times V \to L$ by

$$i(v,u)f = v\,\partial_u f \tag{63}$$

which is the Clifford product of v and the directional derivative of f along u. It is clear that i is bilinear, so it extends to a map $V \otimes V \to L$, and the image of $g \in V \otimes V$ is the Dirac operator.

In an arbitrary basis, let[1] $g = g^{\mu\nu} e_\mu \otimes e_\nu \in V \otimes V$, then we have

$$\begin{aligned} Df &= g^{\mu\nu} i(e_\mu \otimes e_\nu) f \\ &= g^{\mu\nu} e_\mu \frac{\partial f}{\partial x^\nu} \\ &= e^\nu \frac{\partial f}{\partial x^\nu} \end{aligned} \tag{64}$$

where we have defined $e^\nu = g^{\mu\nu} e_\mu$.

Definition 25. Suppose M is a left $\mathrm{Cl}(V)$-module. A real differentiable function $f : U \to M$ defined on an open set $U \subseteq V$ is said to be monogenic at $x \in U$ if $Df(x) = 0$. If f is monogenic at all points in U, then we simply say f is monogenic.

Similarly, if M' is a right $\mathrm{Cl}(V)$-module, we can apply the Dirac operator on the right

$$(fD) = \frac{\partial f}{\partial x^\nu} e^\nu \tag{65}$$

and define monogenic functions with values in M'. We can regard $\mathrm{Cl}(V)$ itself as a $\mathrm{Cl}(V)$-module and speak of left or right monogenic functions $f : V \to \mathrm{Cl}(V)$ depending on whether we are treating $\mathrm{Cl}(V)$ as a left or right module.

[1] Einstein summation convention is in force. Two repeated indices are summed over.

4. Vahlen Matrices

The relation between conformal transformations and Vahlen matrices, which are certain 2×2 matrices in Clifford algebra, is well known. We provide a review of the connection here. For detailed expositions, readers may refer to [1, 7].

4.1. Algebra Isomorphism $\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})\cong \mathrm{Mat}(2,\mathrm{Cl}(V))$.

The key ingredient is the $(1,1)$ periodicity of Clifford algebra (see [4] for details), which is the following proposition.

Proposition 26. *For a vector space V with signature (p,q), the Clifford algebra $\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})$ is isomorphic to the matrix algebra $\mathrm{Mat}(2,\mathrm{Cl}(V))$.*

Proof. An explicit isomorphism $i:\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})\to \mathrm{Mat}(2,\mathrm{Cl}(V))$ can be defined by extending

$$i(x)=\begin{pmatrix}x&0\\0&-x\end{pmatrix},\quad i(e_-)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad i(e_+)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tag{66}$$

for vectors $x\in V$. Using linearity, it suffices to check that $i(u)i(v)+i(v)i(u)=-2g(u,v)$ for all $u,v\in V\cup\{e_-,e_+\}$.

(1) When $u,v\in V$, we have

$$\begin{pmatrix}u&0\\0&-u\end{pmatrix}\begin{pmatrix}v&0\\0&-v\end{pmatrix}+\begin{pmatrix}v&0\\0&-v\end{pmatrix}\begin{pmatrix}u&0\\0&-u\end{pmatrix}=\begin{pmatrix}uv+vu&0\\0&uv+vu\end{pmatrix}=-2g(u,v).\tag{67}$$

(2) When $u\in V$ and $v=e_\mp$, we have

$$\begin{pmatrix}u&0\\0&-u\end{pmatrix}\begin{pmatrix}0&\pm1\\1&0\end{pmatrix}+\begin{pmatrix}0&\pm1\\1&0\end{pmatrix}\begin{pmatrix}u&0\\0&-u\end{pmatrix}=\begin{pmatrix}0&\pm u\\-u&0\end{pmatrix}+\begin{pmatrix}0&\mp u\\u&0\end{pmatrix}=0.\tag{68}$$

(3) When $u=v=e_\mp$, we have

$$\begin{pmatrix}0&\pm1\\1&0\end{pmatrix}^2=\pm1.\tag{69}$$

(4) When $u\in e_+$ and $v=e_-$, we have

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}+\begin{pmatrix}1&0\\0&-1\end{pmatrix}=0.\tag{70}$$

Hence, $i:V\oplus\mathbb{R}^{1,1}\to\mathrm{Mat}(2,\mathrm{Cl}(V))$ induces $i:\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})\to\mathrm{Mat}(2,\mathrm{Cl}(V))$ by universal property. Furthermore, note that for $a\in\mathrm{Cl}(V)\subseteq\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})$, we have

$$i(a)=\begin{pmatrix}a&0\\0&\hat{a}\end{pmatrix}\tag{71}$$

and so for an arbitrary element $a+be_-+ce_++de_-e_+\in\mathrm{Cl}(V\oplus\mathbb{R}^{1,1})$ where $a,b,c,d\in\mathrm{Cl}(V)$, we have

$$\begin{aligned}i(a+be_-+ce_++de_-e_+)&=\begin{pmatrix}a&0\\0&\hat{a}\end{pmatrix}+\begin{pmatrix}b&0\\0&\hat{b}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}+\cdots\\&=\begin{pmatrix}a+d&b-c\\\hat{b}+\hat{c}&\hat{a}-\hat{d}\end{pmatrix}.\end{aligned}\tag{72}$$

We can verify that i is an algebra isomorphism by showing that the inverse is

$$i^{-1}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\frac{1}{2}(a+\hat{d})+\frac{1}{2}(b+\hat{c})e_-+\frac{1}{2}(-b+\hat{c})e_++\frac{1}{2}(a-\hat{d})e_-e_+.\tag{73}$$

□

This isomorphism is also explicitly constructed by Maks [7] but with the Clifford algebra convention that $x^2 = Q(x)$. Starting now, we will use the isomorphism i to identify $\mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$ with $\mathrm{Mat}(2, \mathrm{Cl}(V))$.

Lemma 27. *The three involutions of* $\mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$ *in terms of* $\mathrm{Mat}(2, \mathrm{Cl}(V))$ *are given by the following formulas:*

$$\begin{aligned} &\text{the grade involution:} && \widehat{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \hat{a} & -\hat{b} \\ -\hat{c} & \hat{d} \end{pmatrix}, \\ &\text{the reversal:} && \widetilde{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}, \\ &\text{the Clifford conjugation:} && \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}. \end{aligned} \tag{74}$$

Proof. These can be readily deduced using the isomorphism between $\mathrm{Cl}(V \oplus \mathbb{R}^{1,1})$ and $\mathrm{Mat}(2, \mathrm{Cl}(V))$. □

Definition 28. A matrix $A \in \mathrm{Mat}(2, \mathrm{Cl}(V))$ is called a Vahlen matrix if it is in the Lipschitz group $\Gamma(V \oplus \mathbb{R}^{1,1})$.

The four types of conformal transformations in proposition 5 can be represented by elements in $\Gamma(V \oplus \mathbb{R}^{1,1})$ which in turn correspond to the following Vahlen matrices:

(1) Translation $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$.
(2) Orthogonal transformation $\left(\begin{smallmatrix} a & 0 \\ 0 & \hat{a} \end{smallmatrix}\right)$.
(3) Dilation $\begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$.
(4) Inversion $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We will describe a more systematic way to characterize Vahlen matrices in the next section, and it will be obvious to see that the matrices listed above are indeed Vahlen matrices.

4.2. **Characterizing Vahlen Matrices.** Maks [7] claims a set of criteria for determining Vahlen matrices leveraging corollary 20. We supply a proof of Maks' criteria in this section.

Proposition 29. *If* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}(2, \mathrm{Cl}(V))$*, then* $A \in \Gamma(V \oplus \mathbb{R}^{1,1})$ *if and only if*

(1) $a\bar{a},\ b\bar{b},\ c\bar{c},\ d\bar{d} \in \mathbb{R}$,
(2) $a\bar{c},\ b\bar{d} \in V$,
(3) $ax\bar{b} - bx\bar{a},\ cx\bar{d} - dx\bar{c} \in \mathbb{R}$ *for all* $x \in V$,
(4) $ax\bar{d} - bx\bar{c} \in V$ *for all* $x \in V$,
(5) $a\tilde{b} = b\tilde{a}, \quad c\tilde{d} = d\tilde{c}$, *and*
(6) *the pseudo-determinant defined as* $\Delta(A) = a\tilde{d} - b\tilde{c}$ *is a nonzero real number.*

Proof. By corollary 20, it is sufficient to show that this list of conditions is equivalent to $A\bar{A} \in \mathbb{R}^\times$ and $Ax\tilde{A} \in V \oplus \mathbb{R}^{1,1}$ for all $x \in V \oplus \mathbb{R}^{1,1}$. By linearity, $Ax\tilde{A} \in V \oplus \mathbb{R}^{1,1}$ for all $x \in V \oplus \mathbb{R}^{1,1}$ is equivalent to the same statement for all $x \in V \cup \{n_0, n_\infty\}$. Recall that n_0 and n_∞ form a basis for $\mathbb{R}^{1,1}$ and they are given by

$$n_0 = \frac{e_- + e_+}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad n_\infty = \frac{e_- - e_+}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{75}$$

Direct calculation shows

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} b\tilde{d} & b\bar{b} \\ d\tilde{d} & d\bar{b} \end{pmatrix},\\
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} a\bar{c} & a\bar{a} \\ c\bar{c} & c\bar{a} \end{pmatrix},\\
&\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix} = \begin{pmatrix} ax\bar{d} - bx\bar{c} & ax\bar{b} - bx\bar{a} \\ cx\bar{d} - dx\bar{c} & cx\bar{b} - dx\bar{a} \end{pmatrix}.
\end{aligned} \tag{76}$$

Demanding all expressions to be vectors for all $x \in V$ is equivalent to the first four criteria. From the calculation

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix}\\
&= \begin{pmatrix} a\tilde{d} - b\tilde{c} & -a\tilde{b} + b\tilde{a} \\ c\tilde{d} - d\tilde{c} & -\tilde{c}b + \tilde{a}d \end{pmatrix},
\end{aligned} \tag{77}$$

we see that $A\bar{A} \in \mathbb{R}^\times$ is equivalent to the last two criteria. □

Recall the definition of the pin group (56). The computation (77) immediately implies the following description of $\mathrm{Pin}(V \oplus \mathbb{R}^{1,1})$.

Corollary 30. *The pin group* $\mathrm{Pin}(V \oplus \mathbb{R}^{1,1})$*, as a subset of* $\mathrm{Mat}(2, \mathrm{Cl}(V))$*, consists of Vahlen matrices* A *with the pseudo-determinant* $\Delta(A) = a\tilde{d} - b\tilde{c} = \pm 1$*.*

At this point, we would like to mention that Cnops [3] has a more refined set of criteria for a matrix $A \in \mathrm{Mat}(2, \mathrm{Cl}(V))$ to be in the Lipschitz group $\Gamma(V \oplus \mathbb{R}^{1,1})$. Cnops' criteria reduces to Ahlfors' criteria in [1] for when V has positive definite signature.

4.3. **Conformal Space.** In order to relate Vahlen matrices to conformal transformations, we observe that the conformal embedding eq. (10) can be rewritten as

$$x \mapsto \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -x \end{pmatrix}. \tag{78}$$

This particular representation encourages us to reinterpret the twisted adjoint action of Lipschitz group as an action on spinor-like objects with two components. We will describe the construction of conformal space made by Maks [7] in which these spinor-like objects live.

Definition 31. The pre-conformal space W_{pre} is the set of products $\{Ae \,|\, \text{Vahlen matrix } A\}$, where

$$e = \frac{1 + e_- e_+}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{79}$$

The group of Vahlen matrices $\Gamma(V \oplus \mathbb{R}^{1,1})$ acts on W_{pre} by multiplication on the left.

We see that any element in the pre-conformal space must have the matrix representation

$$\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \tag{80}$$

with entries $x, y \in \mathrm{Cl}(V)$ satisfying $x\bar{x},\ y\bar{y} \in \mathbb{R}$ and $x\bar{y} \in V$, by proposition 29. Simplifying the notations, we drop the right column and write (x, y) or $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ for eq. (80).

To go from the pre-conformal space to the null cone $\mathcal{N}$ of $V \oplus \mathbb{R}^{1,1}$ as in eq. (78), we can define the map $\gamma : W_{\text{pre}} \to \mathcal{N}$ as

$$\gamma(Ae) = Aen_\infty \widetilde{Ae} = An_\infty \tilde{A}. \tag{81}$$

Equivalently, we can apply lemma 27 and write it in matrix form

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \bar{y} & \bar{x} \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & y\bar{x} \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & -x\bar{y} \end{pmatrix}. \tag{82}$$

The purpose of the map γ is to eventually induce a bijection between the forthcoming conformal space W and the conformal compactification $N = N(V)$ of V. To accomplish this endeavor, let us begin with the following observation.

Lemma 32 (Witt's extension theorem)**.** *Let U be a finite-dimensional vector space (over $\mathbb{R}$) together with a non-degenerate symmetric bilinear form. If $\phi : U_1 \to U_2$ is an isometric isomorphism of two subspaces $U_1, U_2 \subseteq U$, then ϕ extends to an isometric isomorphism $\hat{\phi} : U \to U$.*

Proof. See Lam [6]. □

Lemma 33. *The map $\pi \circ \gamma : W_{\text{pre}} \to N$, which by abuse of notation we will also denote by γ, is surjective.*

Proof. By Witt's extension theorem, for every null vector $x \in \mathcal{N}$, there exists an orthogonal transformation that takes n_∞ to x. Since the twisted adjoint action σ is surjective onto the orthogonal group $O(V \oplus \mathbb{R}^{1,1})$, there exists a Vahlen matrix A such that $\sigma_A(n_\infty) = x$. By eq. (81) and eq. (53), we have

$$\gamma(Ae) = An_\infty \tilde{A} \sim An_\infty \hat{A}^{-1} = \sigma_A(n_\infty) = x. \tag{83}$$

Therefore, it becomes an equality $\gamma(Ae) = x$ in N, and so $\gamma : W_{\text{pre}} \to N$ is surjective. □

Lemma 34. *Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a Vahlen matrix. If $An_\infty \tilde{A}$ is proportional to n_∞, then $c = 0$ and $a, d \in \Gamma(V)$.*

Proof. Since we have

$$n_\infty \sim An_\infty \tilde{A} \sim An_\infty \hat{A}^{-1}, \tag{84}$$

we can let $An_\infty \hat{A}^{-1} = rn_\infty$ where $r \in \mathbb{R}^\times$. Thus, we obtain

$$\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = An_\infty = rn_\infty \hat{A} = r\begin{pmatrix} -\hat{c} & \hat{d} \\ 0 & 0 \end{pmatrix}. \tag{85}$$

As a result, we can conclude that $c = 0$ and $a = r\hat{d}$. Applying proposition 29, we know $\Delta(A) = a\bar{a}/r \in \mathbb{R}^\times$ and $ax\tilde{a}/r \in V$ for all $x \in V$. By corollary 20, we conclude that $a \in \Gamma(V)$ and similarly $d \in \Gamma(V)$. □

Lemma 35. *Let $A_1 = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ be two Vahlen matrices. Then $\gamma(A_1e)$ and $\gamma(A_2e)$ are proportional if and only if there exists $t \in \Gamma(V)$ such that $(a_1, c_1) = (a_2t, c_2t)$.*

Proof. Suppose $(a_1, c_1) = (a_2t, c_2t)$ for some $t \in \Gamma(V)$, then calculation shows that

$$\gamma(A_1e) = \begin{pmatrix} a_1\bar{c}_1 & a_1\bar{a}_1 \\ c_1\bar{c}_1 & -a_1\bar{c}_1 \end{pmatrix} = t\bar{t}\begin{pmatrix} a_2\bar{c}_2 & a_2\bar{a}_2 \\ c_2\bar{c}_2 & -a_2\bar{c}_2 \end{pmatrix} = t\bar{t}\gamma(A_2e). \tag{86}$$

Since $t\bar{t} \in \mathbb{R}^\times$, we conclude that $\gamma(A_1e)$ and $\gamma(A_2e)$ are proportional.

Conversely, suppose $\gamma(A_1 e) = r\gamma(A_2 e)$ for some $r \in \mathbb{R}^\times$. Then, we have

$$A_1 n_\infty \tilde{A}_1 = r A_2 n_\infty \tilde{A}_2, \tag{87}$$

which implies $A_2^{-1} A_1 n_\infty \widetilde{A_2^{-1} A_1} = r n_\infty$. Let $A_2^{-1} A_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we know from lemma 34 that $c = 0$ and $a \in \Gamma(V)$, and thus

$$A_2^{-1} A_1 e = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix} = ea. \tag{88}$$

In other words, $A_1 e = A_2 e a$, or in matrix form

$$\begin{pmatrix} a_1 & 0 \\ c_1 & 0 \end{pmatrix} = \begin{pmatrix} a_2 a & 0 \\ c_2 a & 0 \end{pmatrix}. \tag{89}$$

This finishes the proof. □

This lemma reveals exactly the condition of equivalence on W_{pre} to mirror the passage from $\mathcal{N}$ to N. Hence, we make the following definition of conformal space.

Definition 36. The conformal space W of V is the pre-conformal space W_{pre} modulo the relation that (x_1, y_1) and (x_2, y_2) are equivalent if and only if there exists a $t \in \Gamma(V)$ such that $(x_1, y_1) = (x_2 t, y_2 t)$.

Theorem 37. *The map $\gamma : W \to N$ is well defined and is a bijection. Moreover, the actions by Vahlen matrices commute with γ, that is $\gamma(AX) = \sigma_A(\gamma(X))$ for all Vahlen matrices A and all $X \in W$.*

Proof. By lemma 35, γ is well defined and is injective, and from lemma 33, we know γ is surjective.

The action of a Vahlen matrix A commutes with γ because, for an arbitrary element in W represented by $X \in W_{\text{pre}}$, we have

$$\gamma(AX) = AX n_\infty \tilde{X}\tilde{A} \sim \sigma_A(X n_\infty \tilde{X}) = \sigma_A(\gamma(X)) \tag{90}$$

which becomes an equality in N. □

4.4. **Classification of Points in W.** Recall that eq. (78) was our inspiration for conformal space, so we wish to identify $x \in V$ with $(x, 1) \in W$, but we first have to show that $(x, 1) \in W$. Indeed, $A = \left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right)$ satisfies the conditions of proposition 29, thereby is a Vahlen matrix. Geometrically, A is the composition of translation $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ and inversion $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and so $\gamma(Ae) = A n_\infty \tilde{A}$ is inverting n_∞ to n_0 and translating it by x.

Maks [7] classifies the points (x, y) in W into three classes.

(1) $y\bar{y} \neq 0$. In this case, we have $y^{-1} = \bar{y}/(y\bar{y})$ and

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\bar{y} & x\bar{x} \\ y\bar{y} & -x\bar{y} \end{pmatrix} \sim \begin{pmatrix} xy^{-1} & x\bar{x}(y\bar{y})^{-1} \\ 1 & -xy^{-1} \end{pmatrix} = \gamma\begin{pmatrix} xy^{-1} \\ 1 \end{pmatrix}. \tag{91}$$

Therefore, (x, y) is identified with $(xy^{-1}, 1)$ in W and in turn with xy^{-1} in V.

(2) $y\bar{y} = 0$ and $x\bar{x} \neq 0$. In this case, we have $x^{-1} = \bar{x}/(x\bar{x})$ and

$$\gamma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y\bar{x} & x\bar{x} \\ y\bar{y} & y\bar{x} \end{pmatrix} \sim \begin{pmatrix} -yx^{-1} & 1 \\ y\bar{y}(x\bar{x})^{-1} & yx^{-1} \end{pmatrix} = \gamma\begin{pmatrix} 1 \\ yx^{-1} \end{pmatrix}. \tag{92}$$

Therefore, (x, y) is identified with $(1, yx^{-1})$ in W. The Vahlen matrix that represents inversion is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From which, we recognize $(1, yx^{-1})$ as the inversion of $(yx^{-1}, 1)$.

Thus, (x, y) represents the inversion of yx^{-1}, and since yx^{-1} is a null vector, its inversion does not belong to V.

(3) $x\bar{x} = y\bar{y} = 0$. This part is empty in the Euclidean cases $p = 0$ or $q = 0$. Otherwise, (x, y) represents the limiting point of the null-line generated by $x\bar{y}$.

In light of proposition 6, every conformal transformation $U \to \mathbb{R}^{p,q}$ can be described by Vahlen matrix acting through

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = (ax + b)(cx + d)^{-1} \tag{93}$$

where $cx + d$ is invertible for all $x \in U$. Whenever we write a Vahlen matrix acting on x in this fashion, we always assume that $cx + d$ is invertible.

A relevant result about $cx + d$ that we need to discuss before we go into the next section is that $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}$. More generally, we have the following result.

Lemma 38. *If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix, then $\tilde{a}a, \tilde{b}b, \tilde{c}c, \tilde{d}d \in \mathbb{R}$.*

Proof. Applying corollary 21, we know that

$$\bar{A}n_\infty A = \begin{pmatrix} \tilde{d} & -\tilde{b} \\ -\tilde{c} & \tilde{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \tilde{d}c & -\tilde{d}d \\ -\tilde{c}c & -\tilde{c}d \end{pmatrix} \tag{94}$$

is in $V \oplus \mathbb{R}^{1,1}$, which implies that $\tilde{c}c, \tilde{d}d \in \mathbb{R}$. Similar calculation shows that $\bar{A}n_0 A \in V \oplus \mathbb{R}^{1,1}$ leads to $\tilde{a}a, \tilde{b}b \in \mathbb{R}$. □

Corollary 39. *If $x \in V$, the product $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}$.*

Proof. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+b & a \\ cx+d & c \end{smallmatrix}\right)$ is a Vahlen matrix, $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}$ follows from the fact that $cx + d$ is the an entry of a Vahlen matrix. □

Consequently, when $cx + d$ is invertible, we have $(x\tilde{c} + \tilde{d})(cx + d) \in \mathbb{R}^\times$. It is worth mentioning that Maks [7] claims an even stronger result that if an entry in a Vahlen matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible, then that entry is in $\Gamma(V)$.

5. Conformal Invariance of Monogenic Functions

Monogenic function might not stay monogenic under translations by conformal transformations. That is, starting with a Vahlen matrix A and a function f monogenic at Ax, the composition function $f(Ax)$ need not be monogenic at x. On the other hand, it is well known that in the positive definite case (see [9]), the function

$$J_A(x) f(Ax) = \frac{(cx + d)^{-1}}{|(x\tilde{c} + \tilde{d})(cx + d)|^{n/2-1}} f(Ax) \tag{95}$$

where $a = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is monogenic at x whenever f is monogenic at Ax. We prove that this statement also holds when the underlying vector space V has a quadratic form Q with mixed signature.

Since we have

$$e^\mu \frac{\partial}{\partial x^\mu}(J_A(x) f(Ax)) = e^\mu \frac{\partial J_A}{\partial x^\mu} f(Ax) + e^\mu J_A(x) \frac{\partial (Ax)^\nu}{\partial x^\mu} \frac{\partial f}{\partial x^\nu}, \tag{96}$$

the proof consists of two steps. The first step is to find (or verify) the form of J_A such that

$$e^\mu J_A(x) \frac{\partial (Ax)^\nu}{\partial x^\mu} = (\text{some function of } x) e^\nu, \tag{97}$$

so that the second term vanishes by monogenicity of f. Then, we normalize J_A so that J_A is left monogenic, making the first term vanish. The central idea is that $\frac{\partial(Ax)^\nu}{\partial x^\mu}$ is going to be some orthogonal transformation and dilation, so J_A can be some element in the Lipschitz group that undoes the change. This concept is similar to the calculation done in eq. (135) in the appendix, where we demonstrate that the Dirac operator commutes with a certain form of pullback.

Lemma 40. *Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix, then*

$$Ax - Ay = \Delta(A)(y\tilde{c} + \tilde{d})^{-1}(x-y)(cx+d)^{-1} \tag{98}$$

for all x, y such that $cx + d$ and $cy + d$ are invertible.

Proof. Direct calculation is possible (see [1, 8]). Nevertheless, let us apply induction to the fact that every conformal transformation is a finite composition of translation, reflection, and dilation, and inversion. It is straightforward to show that the formula holds for translation, reflection, and dilation, and inversion. We simply show that if the formula is true for A_1 and A_2, then the formula holds for $A_{21} = A_2A_1$, that is we need to show

$$\begin{aligned} &\Delta(A_2)\Delta(A_1)((A_1y)\tilde{c}_2 + \tilde{d}_2)^{-1}(y\tilde{c}_1 + \tilde{d}_1)^{-1}(x-y)(c_1x+d_1)^{-1}(c_2(A_1x)+d_2)^{-1} \\ &\quad = \Delta(A_{21})(y\tilde{c}_{21} + \tilde{d}_{21})^{-1}(x-y)(c_{21}x+d_{21})^{-1}. \end{aligned} \tag{99}$$

Since $\Delta(A_2)\Delta(A_1) = A_2\bar{A}_2A_1\bar{A}_1 = A_2(A_1\bar{A}_1)\bar{A}_2 = \Delta(A_{21})$, it remains to show

$$\begin{aligned} ((A_1y)\tilde{c}_2 + \tilde{d}_2)^{-1}(y\tilde{c}_1 + \tilde{d}_1)^{-1} &= (y\tilde{c}_{21} + \tilde{d}_{21})^{-1} \\ (c_1x+d_1)^{-1}(c_2(A_1x)+d_2)^{-1} &= (c_{21}x+d_{21})^{-1}. \end{aligned} \tag{100}$$

We can argue by symmetry, so we only need to show

$$c_{21}x + d_{21} = (c_2a_1 + d_2c_1)x + (c_2b_1 + d_2d_1) = (c_2(A_1x)+d_2)(c_1x+d_1), \tag{101}$$

but this is obvious after we recognize that $(A_1x)(c_1x+d_1) = a_1x + b_1$. □

Equation (98) can also be expressed as

$$Ax - Ay = \Delta(A)(x\tilde{c} + \tilde{d})^{-1}(x-y)(cy+d)^{-1} \tag{102}$$

by applying the reversal.

Corollary 41. *The derivative of a conformal transformation can be expressed by*

$$\begin{aligned} \frac{\partial(Ax)}{\partial x^\mu} &= \Delta(A)(x\tilde{c} + \tilde{d})^{-1}e_\mu(cx+d)^{-1} \\ &= \frac{\Delta(A)}{(x\tilde{c} + \tilde{d})(cx+d)}(cx+d)e_\mu(cx+d)^{-1}. \end{aligned} \tag{103}$$

Proof. The calculation is a simple application of the product rule. We obtain

$$\frac{\partial(Ax)}{\partial x^\mu} = \left.\frac{d(A(x+he_\mu) - Ax)}{dh}\right|_{h=0} = \Delta(A)(x\tilde{c} + \tilde{d})^{-1}e_\mu(cx+d)^{-1}. \tag{104}$$

□

This allows us to verify directly that $x \mapsto Ax$ is a conformal transformation (compare with eq. (4)).

Lemma 42. *We have:*

$$A^* g = \Omega_A^2 g, \tag{105}$$

where the conformal factor is

$$\Omega_A(x) = \frac{\Delta(A)}{(x\tilde{c} + \tilde{d})(cx+d)}. \tag{106}$$

Proof. Rearranging eq. (103), we obtain

$$\frac{(x\tilde{c} + \tilde{d})(cx+d)}{\Delta(A)} \frac{\partial (Ax)^\nu}{\partial x^\mu} e_\nu = (cx+d) e_\mu (cx+d)^{-1}. \tag{107}$$

By proposition 11, we have

$$g_{\mu\lambda} = (cx+d) \left(\frac{-e_\mu e_\lambda - e_\lambda e_\mu}{2} \right) (cx+d)^{-1}. \tag{108}$$

We can insert $(cx+d)^{-1}(cx+d)$ between e_μ and e_λ and apply eq. (107) to obtain

$$g_{\mu\lambda} = \left[\frac{(x\tilde{c} + \tilde{d})(cx+d)}{\Delta(A)} \right]^2 \frac{\partial (Ax)^\nu}{\partial x^\mu} \left(\frac{-e_\nu e_\gamma - e_\gamma e_\nu}{2} \right) \frac{\partial (Ax)^\gamma}{\partial x^\lambda}. \tag{109}$$

Rearrangement yields the desired result

$$\frac{\partial (Ax)^\nu}{\partial x^\mu} g_{\nu\gamma} \frac{\partial (Ax)^\gamma}{\partial x^\lambda} = \left[\frac{\Delta(A)}{(x\tilde{c} + \tilde{d})(cx+d)} \right]^2 g_{\mu\lambda}. \tag{110}$$

□

Similarly, we also have

$$\frac{\partial (Ax)^\nu}{\partial x^\mu} g^{\mu\lambda} \frac{\partial (Ax)^\gamma}{\partial x^\lambda} = \Omega_A(x)^2 g^{\nu\gamma}, \tag{111}$$

and pairing both sides with e_ν, we obtain

$$\frac{\partial (Ax)}{\partial x^\mu} g^{\mu\lambda} \frac{\partial (Ax)^\gamma}{\partial x^\lambda} = \Omega_A(x)^2 e^\gamma. \tag{112}$$

We can substitute eq. (103) and arrive at

$$\Omega_A(x)(cx+d)^{-1} e^\nu = e^\mu (cx+d)^{-1} \frac{\partial (Ax)^\nu}{\partial x^\mu}. \tag{113}$$

This finishes the first step of our proof to find the form of J_A.

To prove the monogenicity of J_A, we apply induction to the fact that every conformal transformation can be written as a finite composition of translation, orthogonal transformation, dilation, and inversion. Let us separate the inductive step and the base cases into two lemmas.

Lemma 43. *Given Vahlen matrices A_1 and A_2, then*

$$J_{A_2 A_1}(x) = J_{A_1}(x) J_{A_2}(A_1 x) \tag{114}$$

whenever both sides are well defined.

Proof. Let $j_A(x) = cx + d$, then we can express J_A as

$$J_A(x) = \frac{(j_A(x))^{-1}}{|\widetilde{j_A(x)} j_A(x)|^{n/2-1}}. \tag{115}$$

We have shown that $j_{A_2A_1}(x) = j_{A_2}(A_1(x)) j_{A_1}(x)$ in eq. (101). Therefore,

$$\begin{aligned} J_{A_2A_1}(x) &= \frac{(j_{A_2A_1}(x))^{-1}}{|\widetilde{j_{A_2A_1}(x)} j_{A_2A_1}(x)|^{n/2-1}} \\ &= \frac{(j_{A_1}(x))^{-1}}{|\widetilde{j_{A_1}(x)} j_{A_1}(x)|^{n/2-1}} \frac{(j_{A_2}(A_1x))^{-1}}{|\widetilde{j_{A_2}(A_1x)} j_{A_2}(A_1x)|^{n/2-1}} \\ &= J_{A_1}(x) J_{A_2}(A_1x). \end{aligned} \tag{116}$$

□

Lemma 44. *When A is a Vahlen matrix that represents a translation, reflection, dilation, or inversion, the function $J_A(x)$ is left monogenic wherever it is defined.*

Proof. For translation, reflection, and dilation, J_A is just a constant function, so it is monogenic. For inversion, we have

$$J_A(x) = \frac{x \operatorname{sgn}(x^2)}{|x^2|^{n/2}}. \tag{117}$$

The following identities can be verified by direct calculation:

$$\frac{\partial x}{\partial x^\nu} = \frac{\partial(x^\mu e_\mu)}{\partial x^\nu} = e_\nu, \quad \frac{\partial(x^2)}{\partial x^\nu} = -2g_{\nu\lambda}x^\lambda, \quad e^\nu e_\nu = g^{\mu\nu} e_\mu e_\nu = -g^{\mu\nu} g_{\mu\nu} = -n. \tag{118}$$

Then, applying the Dirac operator, we obtain

$$\begin{aligned} DJ_A(x) = e^\nu \frac{\partial J_A}{\partial x^\nu} &= \frac{e^\nu e_\nu \operatorname{sgn}(x^2)}{|x^2|^{n/2}} + n\frac{(e^\nu g_{\nu\lambda} x^\lambda)x}{|x^2|^{n/2+1}} \\ &= \frac{1}{|x^2|^{n/2+1}} \left(-nx^2 + nx^2\right) \\ &= 0. \end{aligned} \tag{119}$$

□

Theorem 45. *Suppose V is a vector space with arbitrary non-degenerate signature. For each Vahlen matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with entries in $\mathrm{Cl}(V)$, the function $J_A(x)f(Ax)$ is monogenic at x whenever f is monogenic at Ax.*

Proof. As mentioned before, it suffices to prove that J_A satisfies eq. (97) and is monogenic. Using eq. (113), we obtain

$$\begin{aligned} e^\mu J_A(x) \frac{\partial(Ax)^\nu}{\partial x^\mu} &= \frac{1}{|(x\tilde{c} + \tilde{d})(cx+d)|^{n/2-1}} e^\mu (cx+d)^{-1} \frac{\partial (Ax)^\nu}{\partial x^\mu} \\ &= \frac{\Omega_A(x)(cx+d)^{-1}}{|(x\tilde{c} + \tilde{d})(cx+d)|^{n/2-1}} e^\nu \\ &= \Omega_A(x) J_A(x) e^\nu \end{aligned} \tag{120}$$

which shows that eq. (97) is satisfied.

We can prove the monogenicity of J_A by applying induction to the fact that A can be written as a finite composition of translation, reflection, dilation, and inversion. The base cases are established by lemma 44. For the inductive step, assume J_{A_1} and J_{A_2} are monogenic, we wish to show that $J_{A_1 A_2}$ is monogenic. Using lemma 43 and eq. (120), we obtain

$$\begin{aligned} DJ_{A_1A_2}(x) &= e^\mu \frac{\partial}{\partial x^\mu}(J_{A_1}(x)J_{A_2}(A_1x)) \\ &= e^\mu \frac{\partial J_{A_1}}{\partial x^\mu} f(Ax) + e^\mu J_{A_1}(x)\frac{\partial (A_1x)^\nu}{\partial x^\mu}\frac{\partial J_{A_2}}{\partial x^\nu} \\ &= \Omega_{A_1}(x)J_{A_1}(x)e^\nu \frac{\partial J_{A_2}}{\partial x^\nu} \\ &= 0, \end{aligned} \tag{121}$$

showing that $J_{A_1A_2}(x)$ is monogenic; this proves the inductive step. □

Note that we can obtain a stronger result that for a real differentiable left $\mathrm{Cl}(V)$-module valued function f:

$$D(J_A f(Ax)) = \Omega_A(x)J_A(x)Df(Ax). \tag{122}$$

One can follow a similar procedure and arrive at an analogous result for right monogenic functions.

Proposition 46. *For a real differentiable right* $\mathrm{Cl}(V)$*-module valued function* f*, we have*

$$(f(Ax)\tilde{J}_A)D = (fD)(x)\tilde{J}_A(x)\Omega_A(x) \tag{123}$$

where

$$\tilde{J}_A(x) = \widetilde{J_A(x)} = \frac{(x\tilde{c}+\tilde{d})^{-1}}{|(x\tilde{c}+\tilde{d})(cx+d)|^{n/2-1}}. \tag{124}$$

In particular, the function $f(Ax)J_A(x)$ is right monogenic at x whenever f is right monogenic at Ax.

Appendix A. A Note on the Definition of the Dirac Operator

We introduced Dirac operator with plus and minus signs in eq. (59), and we have shown that this is a natural construction from the point of view of basis independence. Since this phenomenon does not happen in the positive definite case, it is perhaps worthwhile to show that the classical formula eq. (95) fails if Dirac operator were defined differently, for example, having all positive signs. Consider $\mathbb{R}^{2,1}$ with $e_1^2 = -1$, $e_2^2 = -1$, and $e_3^2 = 1$. The ostensible Dirac operator reads

$$\tilde{D} = e_1\frac{\partial}{\partial x^1} + e_2\frac{\partial}{\partial x^2} + e_3\frac{\partial}{\partial x^3}. \tag{125}$$

The function $f(x) = x^1e_1 + x^3e_3$ is in the kernel of $\tilde{D}$, but if we consider the orthogonal transformation represented by the Vahlen matrix

$$A = \begin{pmatrix} \cosh\alpha + e_2e_3\sinh\alpha & 0 \\ 0 & \cosh\alpha + e_2e_3\sinh\alpha \end{pmatrix}, \tag{126}$$

we find that $J_A(x)f(Ax)$ is not in the kernel of $\tilde{D}$. Carrying out the explicit calculation, we have $J_A(x) = \cosh\alpha - e_2e_3\sinh\alpha$ and

$$\begin{aligned} Ae_2 &= e_2\cosh 2\alpha + e_3\sinh 2\alpha, \\ Ae_3 &= e_2\sinh 2\alpha + e_3\cosh 2\alpha. \end{aligned} \tag{127}$$

Therefore, we have

$$f(Ax) = x^1e_1 + (x^3\cosh 2\alpha + x^2\sinh 2\alpha)e_3. \tag{128}$$

However, $J_A(x)f(Ax)$ is not in the kernel of $\tilde{D}$ because

$$\begin{aligned} J_A(x)f(Ax) &= (\cosh\alpha - e_2e_3\sinh\alpha)\big(x^1e_1 + (x^2\sinh 2\alpha + x^3\cosh 2\alpha)e_3\big) \\ &= x^1e_1(\cosh\alpha - e_2e_3\sinh\alpha) + (x^3\cosh 2\alpha + x^2\sinh 2\alpha)(e_3\cosh\alpha - e_2\sinh\alpha) \end{aligned} \tag{129}$$

and $\tilde{D}$ acting on the first term yields

$$\tilde{D}[x^1e_1(\cosh\alpha - e_2e_3\sinh\alpha)] = -(\cosh\alpha - e_2e_3\sinh\alpha), \tag{130}$$

whereas the second term results in

$$\tilde{D}[(x^3\cosh 2\alpha + x^2\sinh 2\alpha)(e_3\cosh\alpha - e_2\sinh\alpha)] = \cosh 3\alpha + e_2e_3\sinh 3\alpha. \tag{131}$$

They clearly do not cancel with each other. To contrast, if we use the true Dirac operator, we would have

$$D[(x^3\cosh 2\alpha + x^2\sinh 2\alpha)(e_3\cosh\alpha - e_2\sinh\alpha)] = \cosh\alpha - e_2e_3\sinh\alpha \tag{132}$$

and still $D[x^1e_1(\cosh\alpha - e_2e_3\sinh\alpha)] = -(\cosh\alpha - e_2e_3\sinh\alpha)$, which is desired.

More broadly, the true Dirac operator plays well with orthogonal transformations, while the ostensible Dirac operator does not. To be more precise, consider orthogonal transformation $\phi : V \to V$, it induces a Clifford algebra homomorphism $\phi : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$. Explicitly, if a is a Lipschitz group element that represents ϕ, then $\phi : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ is given by

$$\phi(x) = \sigma_a(x) = \hat{a}xa^{-1}. \tag{133}$$

Therefore, there is a natural pullback operation for functions $f : V \to \mathrm{Cl}(V)$ defined by

$$\phi^* f = a^{-1}(f \circ \phi)\hat{a}. \tag{134}$$

The true Dirac operator satisfies $D(\phi^* f) = \pm\phi^* Df$ where the sign depends on whether ϕ is orientation preserving or reversing. This is because

$$\begin{aligned} D\phi^* f &= e^\mu \frac{\partial\phi^\nu}{\partial x^\mu} a^{-1} \frac{\partial f}{\partial x^\nu}\hat{a} \\ &= a^{-1}e^\mu \hat{a} a^{-1} \frac{\partial f}{\partial x^\nu}\hat{a} \\ &= (\hat{a}a^{-1})\phi^* Df. \end{aligned} \tag{135}$$

On the other hand, the ostensible Dirac operator does not have such property as the previous example shows.

Another aspect where the ostensible Dirac operator is not well behaved is that the J_A factor for inversion is not in the kernel of $\tilde{D}$. Simply consider $\mathbb{R}^{1,1}$ with $e_1^2 = -1$, $e_2^2 = 1$. For inversion, we have

$$J_A(x) = \frac{(x^1 e_1 + x^2 e_2)\,\mathrm{sgn}((x^1)^2 - (x^2)^2)}{|(x^1)^2 - (x^2)^2|}. \tag{136}$$

Since $\tilde{D}[x^1 e_1 + x^2 e_2] = e_1 e_1 + e_2 e_2 = 0$, we have

$$\begin{aligned} \tilde{D} J_A(x) &= \frac{(2x^1 e_1 - 2x^2 e_2)(x^1 e_1 + x^2 e_2)}{|(x^1)^2 - (x^2)^2|^2} \\ &= 2 \cdot \frac{-(x^1)^2 - (x^2)^2 + 2x^1 x^2 e_1 e_2}{|(x^1)^2 - (x^2)^2|^2}, \end{aligned} \tag{137}$$

which is clearly nonzero.

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