SPECTRAL SIMPLICITY IN QUANTUM GRAPHS

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Abstract. Given a finite connected graph Γ with n edges, for what edge lengths $\vec{\ell} = (\ell_1, \ldots, \ell_n)$ is the quantum graph $(\Gamma, \vec{\ell}, \Delta)$ spectrally simple? Leonid Friedlander, in his 2005 paper [4], proved for generic $\vec{\ell}$, a non-circle quantum graph is spectrally simple. Later, Yves Colin de Verdière in [3] established a relationship between degenerate eigenvalues of a quantum graphs and singularities of its corresponding secular manifold. He also posed the following question: *when does the secular manifold of a quantum graph admit a singular set of codimension at least two?*

Applying tools from algebraic geometry, we answer Colin de Vedière's codimension 2 question for star graphs in the affirmative; the secular manifold admits a singular set of codimension at least 2. We outline a method to answer the same question in arbitrary quantum graphs, given that they are loop-free and non-mandarin.

1. Introduction

A quantum graph is a mathematical model for common physical phemonena. The classical example is that of an electric circuit. The physical circuit is modeled by a *metric graph* $(\Gamma, \vec{\ell})$ where Γ is a finite connected topological graph, whose n edges are assigned positive lengths $\vec{\ell} = (\ell_1, \ldots, \ell_n)$. Real valued functions on $(\Gamma, \vec{\ell})$ are representative of electric currents on the circuit. They are defined edge-wise by functions on the intervals $[0, \ell_i]$. Differentiation is understood along each edge, which extends to an edge-wise notion of differentiation for functions on the metric graph $(\Gamma, \vec{\ell})$. Define the Laplacian operator Δ on these functions by

$$\Delta(f) = \Delta(\{f_i : i = 1, \ldots, n\}) = \left\{ -\frac{d^2 f_i}{dx^2} : i = 1, \ldots, n \right\}.$$

The tuple $(\Gamma, \vec{\ell}, \Delta)$ is called a *quantum graph*.

The *spectrum* of a quantum graph refers to eigenvalues of the Laplacian Δ, which are necessarily non-negative [2]. The spectrum is dependent on the type of boundary conditions imposed; for the purposes of this paper, *Neumann* and *Kirchhoff* conditions are enforced at the vertices (see section 1). It is often preferred to work with square roots k of eigenvalues k^2 of the graph. The positive square root function is bijective on $\mathbb{R}_{\geq 0}$. In accordance with [1], the spectrum of the quantum graph shall refer to the square roots rather than the eigenvalues themselves. An element of the spectrum is called *simple* if its eigenvalue has multiplicity 1 and *degenerate* otherwise. Similarly, a quantum graph is said to be *spectrally simple* if all of its eigenvalues are simple and *degenerate* otherwise.

We concern ourselves with the following: *given a finite connected graph* Γ*, for what length vectors* $\vec{\ell}$ *is* $(\Gamma, \vec{\ell}, \Delta)$ *spectrally simple?* This question has been addressed multiple times within the literature [1, 3, 4]. The first notable result is due to Friedlander [4]:

> *"Let Γ be a connected metric graph that is different from a circle... Let $[\mathcal{M}_\Gamma]$ be the set in the parameter space $\mathbb{R}^n_+$ of metrics, for which all eigenvalues off are simple. Then the set $[\mathcal{M}_\Gamma]$ is residual."*

As remarked by Alon in [1], while this ensures the density of $\mathcal{M}_\Gamma$, it does not address its measure. In [3], Colin de Verdière expanded on this statement by considering the *secular manifold* Σ_Γ. The secular manifold $\Sigma_\Gamma \subset \mathbb{C}^n$ is the set of n-tuples $\exp(ik\vec{\ell}) = (e^{ik\ell_1}, \ldots, e^{ik\ell_n})$ such that k is in the spectrum of $(\Gamma, \vec{\ell}, \Delta)$ (see [3] and section 1). It turns out that Σ_Γ is the intersection and affine algebraic set V_Γ and the n-torus $\{(z_1, \ldots, z_n) : |z_i| = 1\}$. Colin de Verdière observed that the singular set of Σ_Γ corresponds to n-tuples $\exp(ik\vec{\ell})$ for which k is a degenerate point in the spectrum; using Friedlander's result, he proved that for Γ different than the circle, the dimension of $\Sigma_\Gamma^{\text{sing}}$ is strictly less than that of Σ_Γ (see theorem 1.1 of [3]). This difference, the *codimension*, is defined to be

$$c_\Gamma := \operatorname{codim}(\Sigma_\Gamma^{\text{sing}}, \Sigma_\Gamma) = \dim \Sigma_\Gamma - \dim \Sigma_\Gamma^{\text{sing}}.$$

Colin de Verdière's proof that c_Γ is positive translates to a stronger version of Friedlander's result. More specifically, the complement of $\mathcal{M}_\Gamma$ is a subanalytic set whose codimension is bounded below by c_Γ, which also implies $\mathcal{M}_\Gamma$ being of full measure [1].

It is worth noting that any improvement on the codimension c_Γ yields a stronger genericity result for $\mathcal{M}_\Gamma$. The secular manifold Σ_Γ is the vanishing locus of the secular determinant P_Γ on the n-torus (see section 1). Colin de Verdière conjectured in [3] that P_Γ was reducible only for graphs Γ admitting isometric reflection symmetries for all possible metric graphs on Γ. However, since the edges have variable lengths, reflection symmetries are hard to come by. They occur only if Γ has loops or is mandarin (see section 1). Kurasov and Sarnak later confirmed Colin de Verdière's claim [6]. The bound $c_\Gamma \geq 1$ is optimal when Γ is reducible. Colin de Verdière suggests for loop-free non-mandarin graphs, the codimension c_Γ is at least two [3]. Alon further states this as a conjecture [1]. Colin de Verdière also indicates that singularities of star graphs (see Figure 1) are of interest. In this paper, we prove the following:

Theorem 1. *All star graphs have the property that $c_\Gamma \geq 2$.*

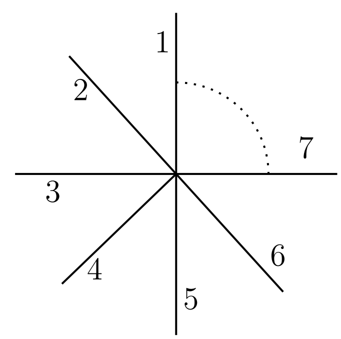

Figure 1: A depiction of a general star graph with 7 or more edges.

The family of star graphs is the first infinite family for which the conjecture has been verified. The dimension of Σ_Γ is $n-1$, where n is the number of edges on the graph Γ. This introduces a number of obvious computational constraints.

Our approach to proving Theorem 1 relies on basic tools from algebraic geometry. The codimension c_Γ is bounded below by codimension of the Zariski closures of Σ_Γ and $\Sigma_\Gamma^{\text{sing}}$. Assuming the graph is loop-free and non-mandarin, a substantial amount of information is known about the vanishing ideal of $\Sigma_\Gamma^{\text{sing}}$. A common practice in algebraic geometry is to take intersections with hyperplanes to obtain dimension bounds. This is not applicable in affine space, so we projectivize the problem and consider the intersection with the hyperplane at infinity. There are reasons to believe this proof extends to the collection of all tree graphs, although this remains to be shown.

1. Secular Manifold: Construction and Convention.

Flexibility of the quantum graph model is due, in part, to the large variety of boundary conditions one may impose. The most relevant to this discussion are as follows:

(i) Neumann conditions. A continuous function $f : \Gamma \to \mathbb{R}$ satisfies *Neumann conditions* at a vertex v of Γ if $f_e(v)$ takes the same value for every edge e incident to v.

(ii) Kirchhoff conditions. A differentiable function $f : \Gamma \to \mathbb{R}$ satisfies *Kirkoff conditions* at a vertex v of Γ if

$$\sum_{e\sim v} \frac{df_e}{dx}(v) = 0.$$

The conditions above are also called *continuity* and *current conditions* respectively. We restrict our attention to functions on Γ which satisfy *(i)* and *(ii)* at every vertex of the graph.

For small graphs Γ with restraints on edge lengths, the spectrum is rather computable. Let's consider the 3-star graph, shown in Figure 2, whose edges all have the same length. An eigenfunction of Δ consists of three twice differentiable edge functions $f_i : [0, \ell_i] \to \mathbb{R}$, satisfying the continuity and current conditions, such that

$$-f_i''(x) = \triangle f_i(x) = k^2 \cdot f_i(x),$$

for some $k \in \mathbb{R}$. Solutions of this differential equation take the form $a_i \sin(kx) + b_i \cos(kx)$ with constants a_i, b_i. The only conditions at the end of each protruding edge are the current conditions, i.e.

$$a_i k = f_i'(0) = 0.$$

Excluding $k = 0$, we arrive at the conclusion that each a_i is zero. The boundary conditions at the center are then

$$b_1 \cos(kL) = b_2 \cos(kL) = b_3 \cos(kL) \quad \& \quad \sum b_i \sin(kL) = 0.$$

Dividing one equation by the next, we obtain a simple expression: $\tan(kL) = 0$. Thus, the eigenvalues of the graph are percisely of the form $(n\pi/L)^2$, for $n \in \mathbb{Z}$.

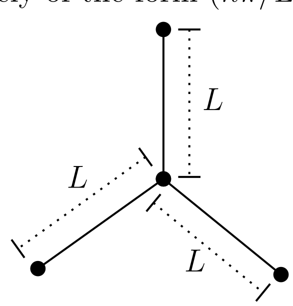

Figure 2.

The above computation was very simple because we set the edge lengths equal. In general, we will rely on a formula which confirms if a pair $(k, \vec{\ell})$ gives an element k of spectrum of

$(\Gamma, \vec{\ell}, \Delta)$ and further if k is degenerate. All this information is packaged into the key definition of this section, the *secular manifold*.

To obtain the desired formula, we need a more systematic approach. For a single edge of Γ and eigenfunction f with $k^2 = \lambda$ of $\triangle$, f_i can be written in a compact exponential form:

$$a_i e^{ikx} + b_i e^{ik(\ell_i - x)} = a_i e^{ikx} + b_i z_i e^{-ikx},$$

where $z_i = e^{ik\ell_i}$ and we number the edges of Γ as $1, \dots, n$. The space of eigenfunctions of $\triangle$ can thus be imagined as an $2n$-tuple $(a_1, \dots, a_n, b_1, \dots, b_n)$. Each vertex of Γ produces a system of equations in a_i, b_i, z_i. With some clever substitutions, see [4], the system is written succinctly as

$$S_\Gamma \cdot D(\vec{z}) \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad \text{where} \quad D(\vec{z}) = \begin{bmatrix} z_1 & & & & & \\ & \ddots & & & 0 & \\ & & z_n & & & \\ & & & z_1 & & \\ & 0 & & & \ddots & \\ & & & & & z_n \end{bmatrix} \tag{1}$$

and where S_Γ is the *secular matrix of* Γ. The rows and columns of the square matrix S_Γ are denoted by $1, \dots, n, \overline{1}, \dots, \overline{n}$, where we simply pick some arbitrary orientation along the graph. According to [2], the entires of S_Γ are determined solely by the graph and are summarized as follows:

$$(S_\Gamma)_{(i,j)} = \begin{cases} \frac{2}{\deg v} - 1 & \text{if } i = \overline{j}, \\ \frac{2}{\deg v} & \text{if } i \text{ follows } j \text{ and } i \neq \overline{j}, \\ 0 & \text{otherwise.} \end{cases} .$$

Equation (1) tells us that k^2-eigenfunctions of $\triangle$ correspond to 1-eigenvectors of $S_\Gamma \cdot D(\vec{z})$, where $D(\vec{z})$ is the diagonal matrix in equation (1). An element k is in the spectrum of $(\Gamma, \vec{\ell}, \Delta)$ only if $\det(I - S_\Gamma \cdot D(\vec{z}))$ evaluates to zero.

For a moment, allow $\vec{z} = (z_1, \dots, z_n)$ to denote symbolic variables and D to denote the same diagonal matrix, now with symbolic variables and no dependence on k nor $\vec{\ell}$. Define

$$P_\Gamma(z_1, \dots, z_n) := \det(I - S_\Gamma \cdot D(z_1, \dots, z_n))$$

as the *secular determinant* of Γ. Its vanishing locus V_Γ in $\mathbb{C}^n$ is called the *secular locus*. Although an arbitrary element $z \in V_\Gamma$ doesn't necessarily have any relevance to the spectrum, if z also lies on the n-torus, then any $(k, \vec{\ell})$ such that $z = \exp(ik\vec{\ell})$ contributes to the spectrum, where $i = \sqrt{-1}$. In particular, the intersection

$$\Sigma_\Gamma := V_\Gamma \cap \mathbb{T}^n,$$

called the *secular manifold*, essentially packages the spectrum of all possible quantum graphs on Γ.

The secular manifold is partitioned into two components

$$\Sigma_\Gamma^{\text{reg}} := \{\vec{z} \in \Sigma_\Gamma : \nabla P_\Gamma(\vec{z}) \neq 0\}$$

$$\Sigma_\Gamma^{\text{sing}} := \{\vec{z} \in \Sigma_\Gamma : \nabla P_\Gamma(\vec{z}) = 0\},$$

called the *regular set* and *singular set* respectively. Our interest in this particular partition is summarized by the following theorem of Colin de Verdière:

Theorem. [Colin de Verdière, 3] *An eigenvalue* k^2 *of the quantum graph* $(\Gamma, \vec{\ell}, \triangle)$ *is a multiple eigenvalue if and only if* $\exp(ik \cdot \vec{\ell})$ *is an element of* $\Sigma_\Gamma^{\text{sing}}$.

Proof. See [3].

Example: Singular Points of $\boldsymbol{\Sigma}_{\textbf{3-star}}$.. In order to determine the singularities, we calculate the secular determinant. Orient the edges inward (see Figure 2). The secular matrix is given in block-form as

$$S_{\text{3-star}} = \begin{bmatrix} 0 & 0 & 0 & -1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 & 2/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 2/3 & 2/3 & -1/3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Each edge is assigned a variable z_1, z_2, z_3. The corresponding secular determinant of the 3-star graph is

$$P_{\text{3-star}} = -z_1^2 z_2^2 z_3^2 - \frac{1}{3} z_1^2 z_2^2 - \frac{1}{3} z_1^2 z_3^2 - \frac{1}{3} z_2^2 z_3^2 + \frac{1}{3} z_1^2 + \frac{1}{3} z_2^2 + \frac{1}{3} z_3^2 + 1.$$

This is an irreducible polynomial. The variety $V_{\text{3-star}}$ has percisely 8 singularities on the torus. They are of the form $(\pm i, \pm i, \pm i)$. Let's restrict our attention on the cube $[-\pi/2, \pi/2]^3$ where the exponential map is a diffeomorphism. The singularities correspond to the corners of the cube. Hence, if all lengths are equal, as in Figure 2, degenerate eigenvalues are of the form $\left(\frac{N\pi}{2L}\right)^2$, with $N \in \mathbb{Z}$ odd.

The singular set $\Sigma_\Gamma^{\text{sing}}$ contains reminant information about length vectors $\vec{\ell}$ giving $(\Gamma, \vec{\ell}, \Delta)$ a degenerate spectrum. Consider the composition

$$F : \mathbb{R}_{\geq 0} \times \mathbb{R}_+^n \xrightarrow{(k,\vec{\ell}) \mapsto k \cdot \vec{\ell}} \mathbb{R}^n \xrightarrow{\exp(i \cdot)} \mathbb{T}^n.$$

The composition F is a submersion if $k > 0$, so one expects the codimension of Σ_Γ and $\Sigma_\Gamma^{\text{sing}}$ to stay fixed after taking preimages. Let π denote projection from $\mathbb{R}_{\geq 0} \times \mathbb{R}_+^n$ to the parameter space of edge lengths. The set $\pi(F^{-1}(\Sigma_\Gamma^{\text{sing}}))$ consists of *all* edge lengths giving degenerate spectrum. Its complement corresponds to the collection of edge lengths $\vec{\ell}$ such that $(\Gamma, \vec{\ell}, \Delta)$ is spectrally simple. Assuming that Σ_Γ admits a regular point, basic notions regarding algebraic sets indicate that

$$\dim \Sigma_\Gamma^{\text{sing}} \leq n - 2,$$

meaning the preimage by F should be codimension 2. After applying the projection, we would then expect the dimension to increase by at most one. That is, informally, we expect the collection of all degenerate lengths to have positive codimension.

A number of issues prevent us from quickly formalizing the argument. To begin, Σ_Γ and its singular set are not necessarily manifolds, as the word *singular* suggests. They do, however, admit a finite stratification by real manifolds, each of whose dimension is bounded by the dimension of its Zariski closure in affine space (see section 2). Additionally, F is not a submersion at $k = 0$. Any pair of the form $(0, \vec{\ell})$ is mapped, by F, to the same point on the torus, namely $p = (1, \ldots, 1)$. If p is an element of $\Sigma_\Gamma^{\text{sing}}$, then indeed, *every* quantum graph on Γ is spectrally degenerate, which is not possible assuming Γ is different from the circle. Thus, one may restrict F to $\mathbb{R}_+^{n+1}$, where it is certainly a submersion. Further, Σ_Γ is never required to admit regular points, meaning our expected dimension bound on $\Sigma_\Gamma^{\text{sing}}$, does not necessarily hold. Finally, projections do not preserve manifold structure. Instead, notions of *subanalytic sets* are required to define dimension.

Alon does confirm our informal expectations; namely, the codimension of $\Sigma_\Gamma^{\text{sing}}$ in Σ_Γ is a lower bound for the codimension of degenerate length vectors $\vec{\ell}$ in $\mathbb{R}^n_+$. Any further improvement on c_Γ then directly corresponds to an improved genericity result for a given topological graph Γ. It is easily checked that the optimal lower bound on c_Γ, given that Γ is not a circle, is 1 (see Figure 3). However, assuming Γ is loop-free and non-mandarin, then Alon and Colin de Verdière suggest that the codimension is at least 2 (see Figure 4). We state this as a conjecture.

Definition. A graph is *mandarin* if it consists of two vertices, at least one edge, and no loops.

Conjecture. *Suppose* Γ *is a loop-free, non-mandarin graph. The codimension* c_Γ *of* $\Sigma_\Gamma^{\text{sing}}$ *in* Σ_Γ *is at least* 2.

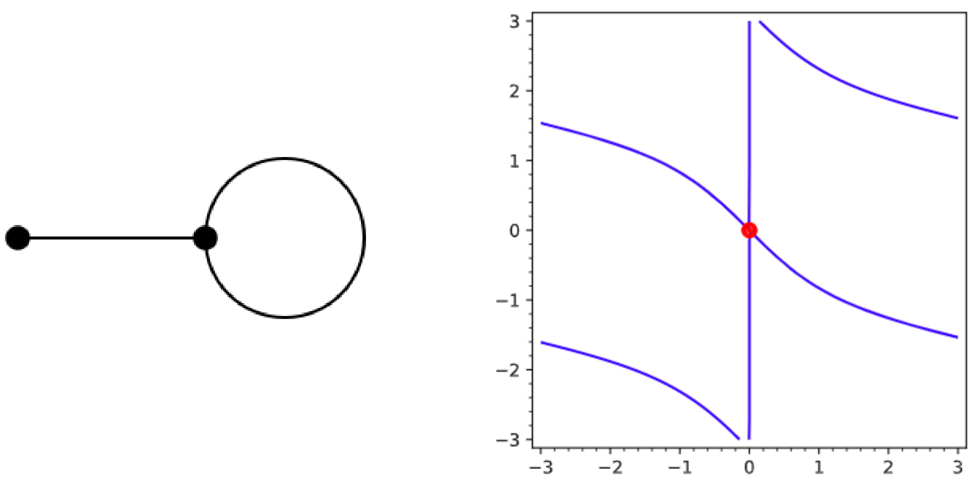

Figure 3: The 'lasso' as a topological graph (left), and its secular manifold (right). The regular points on the secular manifold are shown in blue and the singular points in red.

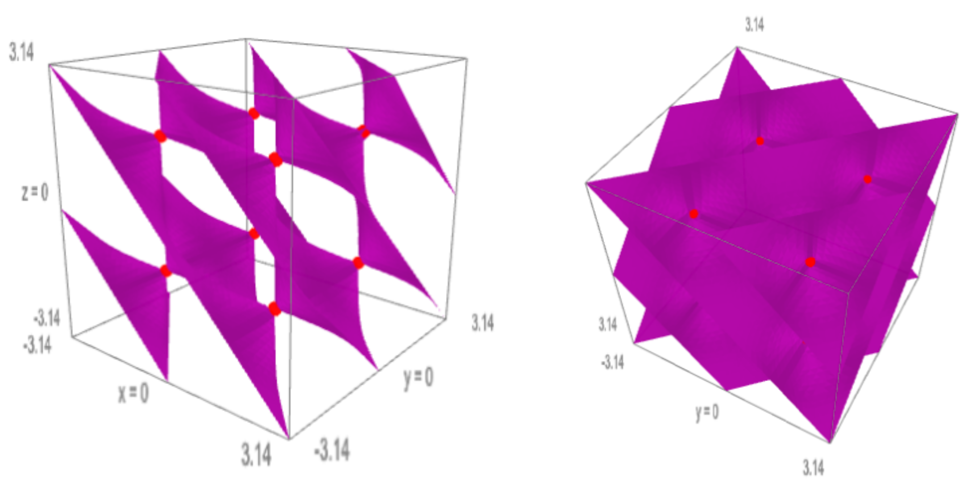

Figure 4: The secular manifold of a 3-star graph, with two different perspectives. The regular points are in purple, while the singular points are highlighted.

2. THE SECULAR LOCUS AND ITS SINGULARITIES.

In this section and the following sections, we will use basic notions from algebraic geometry. Please see Hartshorne's *Algebraic Geometry* [5] for a general reference on the subject.

As briefly mentioned in the previous section, the dimension of Σ_Γ is $n-1$, where n is the number of edges on the graph Γ. We remark, again, that Σ_Γ is not necessarily a manifold. It is a real algebraic set and can admit singular points, where it is not locally Euclidean. A useful trick is to simply delete all singularities, leaving only regular points. Repeating this process gives a stratification of Σ_Γ by manifolds, where the dimension of Σ_Γ is given by its regular points. Here, the terms *regular* and *singular* refer to the geometric usage, rather than the partition of Σ_Γ in the previous section. The dimension computation for Σ_Γ was established by Friedlander and Colin de Verdière, after calculating the tangent space of Σ_Γ at a regular point (see [3, 4]).

It's not clear how to preform a direct calculation for $\dim \Sigma_\Gamma^{\text{sing}}$. Rather, we will obtain upper bounds with Prop. 1, which tells us $\dim \Sigma_\Gamma^{\text{sing}} \leq \dim V$ for every algebraic set V containing $\Sigma_\Gamma^{\text{sing}}$. Naturally, we expect the following inclusions:

$$\Sigma_\Gamma^{\text{reg}} = V_\Gamma^{\text{reg}} \cap \mathbb{T}^n \quad \& \quad \Sigma_\Gamma^{\text{sing}} = V_\Gamma^{\text{sing}} \cap \mathbb{T}^n, \tag{2}$$

where V_Γ^{sing} denotes the singular points of V_Γ as an algebraic set (see Section 5, Chapter I of [5]). It is always true that V_Γ^{sing} has positive codimension in V_Γ, meaning that the same should be true of $\Sigma_\Gamma^{\text{sing}}$ within Σ_Γ. Unfortunately, our partition of Σ_Γ into regular and singular components doesn't always agree with the geometric definition. The circle graph C, as remarked below, is an example of this: the secular manifold of the circle is a regular variety, yet by definition, $\Sigma_C = \Sigma_C^{\text{sing}}$ and $\Sigma_C^{\text{reg}} = \emptyset$.

Proposition 1. [Alon, 1]. *Let V be an algebraic set in $\mathbb{C}^n$, and let M denote the intersection $V \cap \mathbb{T}^n$. The regular points of M form a manifold whose dimension is at most* $\dim V$.

Proof. See [1] or the Appendix.

Remark. *For the circle graph C, $\Sigma_C = \Sigma_C^{\text{sing}}$, and V_C is a regular variety. That is, $V_C^{\text{sing}} \cap \mathbb{T}^1$ is not equal to Σ_C^{sing}. Also, for every length assigned to the circle, its spectrum is degenerate.*

Proof. The secular determinant of the circle C is $P_C = \det\big(I \cdot (1-z)\big) = (1-z)^2$. The vanishing locus of P_C consists of a single point $z = 1$, which lies on the 1-torus. The derivative of P_C is simply $P_C' = 2(z-1)$, so by definition, $\Sigma_C = \{1\} = \Sigma_C^{\text{sing}}$. Simply noting that a point in affine space is a regular variety, we conclude the proof. □

The circle represents a degenerate case in the context of Friedlander's result. However, for non-circle graphs Γ, do the inclusions in (2) hold? The answer is yes (see the corollary below). For an algebraic set $V \subset \mathbb{C}^n$, defined by a single polynomial f, $V^{\text{sing}} = Z(f, \nabla f)$ only if each irreducible factor of f appears with multiplicity 1. In the case of the circle, P_C has a squared factor. This doesn't occur for any other graphs Γ. A factorization of P_Γ by Sarnak and Kurasov confirms this as a property of the secular determinant.

Theorem. [Sarnak & Kurasov, 1, 6] *Let Γ be a quantum graph with n edges. The polynomial $P_\Gamma \in \mathbb{C}[z_1, \ldots, z_n]$ is irreducible if and only if Γ has no reflection symmetries. Moreover, if Γ has a reflection symmetry, then P_Γ factors as follows:*

(1) If Γ has loops, then

$$P_\Gamma = P_{\Gamma,sym} \cdot \prod_{e_j \in \varepsilon_{loops}} (1 - z_j)$$

where ε_{loops} is the set of loops, and $P_{\Gamma,sym}$ is irreducible.

(2) If Γ is a mandarin graph, then

$$P_\Gamma = P_{M,s} \cdot P_{M,as}$$

where both $P_{M,s}$ and $P_{M,as}$ are irreducible multi-linear polynomials of the form

$$P_{M,s} := \sum_{j=1}^{n}(z_j - 1)\prod_{i\neq j}(z_i + 1), \quad P_{M,as} := \sum_{j=1}^{n}(z_j + 1)\prod_{i\neq j}(z_i - 1).$$

Proof. See [6].

Corollary. [Alon, 1]. *Let Γ be a topological graph, which is not a circle. Then $V_\Gamma^{\text{reg}} \cap \mathbb{T}^n = \Sigma_\Gamma^{\text{reg}}$ and $V_\Gamma^{\text{sing}} \cap \mathbb{T}^n = \Sigma_\Gamma^{\text{sing}}$. As a consequence, $\Sigma_\Gamma^{\text{sing}}$ has positive codimension in the secular manifold Σ_Γ.*

Proof. According to the theorem above, as long as $\Gamma \ncong C$, P_Γ factors into irreducible polynomials each with multiplicity 1. Thus, $V_\Gamma^{\text{sing}} = Z(P_\Gamma, \nabla P_\Gamma)$. Recall that

$$\Sigma_\Gamma^{\text{sing}} := \{z \in \Sigma_\Gamma : \nabla P_\Gamma(z) = 0\}.$$

Since Σ_Γ is cut out of the n-torus by P_Γ, the intersection of V_Γ^{sing} with $\mathbb{T}^n$ is necessarily $\Sigma_\Gamma^{\text{sing}}$. The codimension of the singular points of V_Γ is positive. The algebraic set V_Γ is determined by one equation, so $\dim V_\Gamma = n-1$, and $\dim V_\Gamma^{\text{sing}} \leq n-2$. By Prop. 1, the real dimension of $V_\Gamma^{\text{sing}} \cap \mathbb{T}^n$ is at most $n-2$. According to Colin de Verdière's computation, $\dim \Sigma_\Gamma = n-1$. Thus, $\operatorname{codim}(\Sigma_\Gamma^{\text{sing}}, \Sigma_\Gamma)$ is positive. □

Prop. 1 allows us to place the codimension 2 question into algebraic geometry. Consider the following inequality:

$$\operatorname{codim}(\Sigma_\Gamma^{\text{sing}}, \Sigma_\Gamma) = \dim(\Sigma_\Gamma) - \dim(\Sigma_\Gamma^{\text{sing}}) \geq n - 1 - \dim(\overline{\Sigma_\Gamma^{\text{sing}}}),$$

where $\overline{\Sigma_\Gamma^{\text{sing}}}$ denotes the *Zariski closure* of $\Sigma_\Gamma^{\text{sing}}$ (see p.g. 11 of [5]). It therefore suffices to prove that $n-3$ is an upper bound on the dimension of $\overline{\Sigma_\Gamma^{\text{sing}}}$. The corollary above, due to Alon, also gives the inclusion $\overline{\Sigma_\Gamma^{\text{sing}}} \subset V_\Gamma^{\text{sing}}$. Although this is not necessarily a strict inclusion, one can infer about the vanishing ideal of $\overline{\Sigma_\Gamma^{\text{sing}}}$, which could lead to a dimension bound.

For lack of a better term, the *singularity problem* should refer to this question of codimension, and moving forward, we shall say a quantum graph Γ has property $(*)$ if $\Sigma_\Gamma^{\text{sing}}$ has codimension 2 or greater in Σ_Γ.

3. Projectivization of the Singular Locus

Obtaining a dimension bound is a regular occurance in algebraic geometry. Intersection theory provides a means to do so. Consider the classical result: *"If $Y \subset \mathbb{C}^n$ is an affine variety, and V a hypersurface not containing Y, then the dimension of $Y \cap V$ is one less than* $\dim V$*, assuming the intersection is nonempty"* (see Exercise 1.8, Chapter I of [5]). This results suggests that we look how $\overline{\Sigma_\Gamma^{\text{sing}}}$ interacts with affine hypersurfaces. Two critical issues appear. First, one does not expect $\overline{\Sigma_\Gamma^{\text{sing}}}$ to be a variety, so instead, we would need to understand how irreducible components of $\overline{\Sigma_\Gamma^{\text{sing}}}$ interact with hypersurfaces. Secondly, we are required to find hypersurfaces which are not disjoint from $\overline{\Sigma_\Gamma^{\text{sing}}}$. Taken together, these unknowns suggest failure for this line of thinking.

Both issues may be corrected by taking *projective closures*. The projective closure of an affine algebraic set V is its Zariski closure considered as a subset of $\mathbb{P}^n$, where $\mathbb{C}^n$ is identified with one of the standard affine open sets in $\mathbb{P}^n$. The projective closure is usually denoted with a bar notation, i.e. $\overline{V}$. Projective space is ideal for intersection theory. Given some dimension conditions, the intersection of two algebraic sets in projective space is always nonempty. This provides a method to obtain a dimension bound on $\Sigma_\Gamma^{\text{sing}}$, which we state as Prop. 2.

Proposition 2. *Let V be an algebraic set in $\mathbb{P}^n$, and H a hypersurface. If* $\dim V \geq 1$, *then*

$$\dim V \leq \dim V \cap H + 1.$$

Proof. See Appendix.

To projectivize this problem, we start by looking at the homogenization of the secular polynomial P_Γ. The most natural approach is to alter the definition; the *projectivized secular determinant* is

$${}^{\text{proj}}P_\Gamma(z_0, \dots, z_n) = \det(I \cdot z_0 - S_\Gamma \cdot D(z_1, \dots, z_n)),$$

where $I, S, D(\vec{z})$ are as usual and z_0 is the new variable. The polynomial above is homogeneous because the matrix $I \cdot z_0 - S_\Gamma \cdot D(\vec{z})$ is a polynomial matrix whose entires are homogeneous of degree 1. Note that ${}^{\text{proj}}P_\Gamma(1, \vec{z}) = P_\Gamma(\vec{z})$. The projective closure of the secular locus, denoted $\overline{V_\Gamma}$, is called the *projectivized secular locus*. It is the zero locus of ${}^{\text{proj}}P_\Gamma$. The reader should note the following facts regarding projectivization:

(1) If $V \subset \mathbb{C}^n$ is an algebraic set with vanishing ideal $I(V)$, then the vanishing ideal of its projective closure $\overline{V} \subset \mathbb{P}^n$ is generated by homogenizing polynomials in $I(V)$ with respect to z_0. (See Exercise 2.9, Chapter I of [5]).
(2) Projective closure preserves dimension. That is, for an affine algebraic set $V \subset \mathbb{C}^n$, $\dim V = \dim \overline{V}$. (See Exercise 2.6, Chapter I of [5]).

In accordance with Prop. 2, it suffices to find a hypersurface H such that

$$\dim A \cap H \leq n - 4,$$

where A denotes the projective closure of $\Sigma_\Gamma^{\text{sing}}$. If Γ is a *star graph* (see Figure 4), then the bound above will be obtained with H being the *plane at infinity*, or $H = Z(z_0)$. Several key observations regarding star graphs are necessary in order to obtain this conclusion. The proof is given in full rigor in the following section. The resulting theorem, as given in the introduction, is restated below:

Theorem 1. *All star graphs have property* (∗).

4. Application: Star Graphs Satisfy Property (∗).

Fix some star graph Γ with n edges. Some definitions and terminology are required, before we begin the proof. Please refer back to this section for any new terminology.

Definitions & Terminology. A set $V \subset \mathbb{C}^n$ is *symmetric* if it remains fixed under coordinate-wise permutation. So given any point $P = (P_1, \dots, P_n)$ in V and any permutation $\sigma \in S_n$, $\sigma(P) = (P_{\sigma(1)}, \dots, P_{\sigma(n)})$ is also an element of V.

An ideal I of $\mathbb{C}[z_1, \dots, z_n]$ is called *symmetric* if it remains fixed under variable-wise permutation. That is, given any polynomial $f \in I$ and any permutation $\sigma \in S_n$, then $\sigma(f) = f(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ is an element of I.

Let f be a polynomial in variables $z_1, \ldots, z_n$. The polynomial f decomposes as into sum of homogeneous polynomials $f_0 + f_1 + \cdots + f_d$, where $\deg f_k = k$, $f_d \neq 0$, and $\deg f = d$. The *leading homogeneous part of* f is f_d. The *homogenization of* f *by* z_0 is

$$z_0^d f_0 + z_0^{d-1} f_1 + \cdots + z_0 f_{d-1} + f_d.$$

For simplicity, the homogenization of f by z_0 is denoted $\beta(f)$, in accordance with [5].

The *weight of a monomial* is the number of distinct variables in the monomial. For example, $z_1^2 z_2^{10} z_5$ has weight 3 in $\mathbb{C}[z_1, \ldots, z_n]$. The *weight of a polynomial* f is the maximum weight of its terms.

We present four lemmas to aid the proof.

Lemma 1. *A set* $V \subset \mathbb{C}^n$ *is symmetric if and only if its vanishing ideal* $I(V)$ *is symmetric.*

Proof. Assume V is symmetric. Let f be a polynomial in the vanishing ideal of V and σ some element of S_n. Then

$$\sigma(f)(V) = f(\sigma(V)) = f(V) = 0,$$

which confirms that $\sigma(f)$ is an element of $I(V)$, meaning that $I(V)$ is symmetric.

If instead, $I(V)$ is symmetric, find some generators $f_1, \ldots, f_m$ such that V is the vanishing locus $Z(f_1, \ldots, f_m)$. For each f_i and permutation $\sigma \in S_n$, one has

$$f_i(\sigma(V)) = \sigma(f_i)(V) = 0,$$

since $\sigma(f_i)$ is an element of $I(V)$. The generators $f_1, \ldots, f_m$ cut out V as an algebraic set, meaning that the above implies $\sigma(V) = V$ for all $\sigma \in S_n$. That is, V is symmetric. $\square$

Lemma 2. *The algebraic sets* $\overline{\Sigma_\Gamma} \subset \mathbb{C}^n$ *and* $\overline{\Sigma_\Gamma^{\text{sing}}} \subset \mathbb{C}^n$ *are symmetric.*

Proof. The torus $\mathbb{T}^n$ is a symmetric subset of $\mathbb{C}^n$, and we claim that both V_Γ and its singular set are symmetric. If this is true, then the intersections

$$\Sigma_\Gamma = V_\Gamma \cap \mathbb{T}^n \quad \& \quad \Sigma_\Gamma^{\text{sing}} = V_\Gamma^{\text{sing}} \cap \mathbb{T}^n$$

are symmetric sets. The proof of Lemma 1 then tells us that their vanishing ideals are symmetric. The Zariski closures of Σ_Γ and $\Sigma_\Gamma^{\text{sing}}$ respectively have the same vanishing ideal as Σ_Γ and as $\Sigma_\Gamma^{\text{sing}}$. By Lemma 1, the Zariski closures are symmetric algebraic sets.

Now we prove the claim that V_Γ and V_Γ^{sing} are symmetric. The vanishing ideal of V_Γ is generated by P_Γ. The polynomial P_Γ is dependent only on the underlying topological graph of Γ. Permuting the edges causes no change to Γ. Because the edges are associated to variables, P_Γ is fixed under permutation. By Lemma 1, V_Γ is symmetric. Also, for any transposition τ switching z_i, z_j, one has

$$\tau\Big(\frac{\partial P_\Gamma}{\partial z_i}\Big) = \frac{\partial P_\Gamma}{\partial z_j}.$$

Hence, permutations of P_Γ and one of its partials generate the ideal $I = (P_\Gamma, \partial P_\Gamma / \partial z_i)$, defining V_Γ^{sing}. The vanishing ideal $I(V_\Gamma^{\text{sing}})$ is the radical of the symmetric ideal I. Symmetry is a property preserved taking radicals of ideals,[1] so by Lemma 1, we conclude V_Γ^{sing} is symmetric. $\square$

[1]If f^m is an irreducible polynomial contained in the radical of some symmetric ideal J, then every permutation of f belongs to $\sqrt{J}$.

Lemma 3. *All terms of the secular polynomial take the form $z_{i_1}^2 \cdots z_{i_k}^2$. Its leading homogeneous part is a nonzero scalar of $z_1^2 \cdots z_n^2$.*

Proof. The secular determinant of a star graph is very predictable. The secular matrix S_Γ takes the following form:

$$S_\Gamma = \left[\begin{array}{c|c} 0 & M_n \\ \hline I & 0 \end{array}\right], \quad M_n = \frac{1}{n} \cdot \begin{bmatrix} 2-n & 2 & \cdots & \cdots & 2 \\ 2 & 2-n & 2 & \cdots & 2 \\ \vdots & 2 & 2-n & \ddots & \vdots \\ 2 & \cdots & \ddots & \ddots & 2 \\ 2 & \cdots & \cdots & 2 & 2-n \end{bmatrix},$$

where M_n is an $n \times n$ matrix and S_Γ is a $2n \times 2n$ block matrix. Let D_n denote the $n \times n$ matrix consisting of the variables $z_1, \ldots, z_n$ along the diagonal. The formula for the secular determinant P_Γ can be simplified substantially by taking a Schur complement:

$$P_\Gamma = \det \left[\begin{array}{c|c} I & -M_n D_n \\ \hline -D_n & I \end{array}\right] = \det\big(I - (-M_n D_n)(I^{-1})(-D_n)\big) = \det(I - M_n D_n^2).$$

The expression above makes it clear that P_Γ is a polynomial in $z_1^2, \ldots, z_n^2$ since D_n^2 consists of entries in $z_1^2, \ldots, z_n^2$. Further, all z_i-terms in the matrix $I - M_n D_n^2$ belong to the same column, meaning that the terms of f are of the form $z_{i_1}^2 \cdots z_{i_k}^2$.

The leading homogeneous part of P_Γ is equal to ${}^{\text{proj}}P_\Gamma(0, z_1, \ldots, z_n)$; to clarify, it's equal to valuation of its homogenization after substituting $z_0 = 0$. According to the definition given in the previous section, we obtain a formula for the leading homogeneous part of P_Γ:

$${}^{\text{proj}}P_\Gamma(0, z_1, \ldots, z_n) = \det\left(S_\Gamma \cdot \left[\begin{array}{c|c} D_n & 0 \\ \hline 0 & D_n \end{array}\right]\right) = -\det M_n \cdot (\det D_n)^2 = -\det M_n \cdot z_1^2 \cdots z_n^2.$$

The matrix M_n is invertible. To see why, let $\mathbf{2}$ denote the matrix whose entries are all equal to 2. Its only nonzero eigenvalue is $4n$, with corresponding eigenvector $(1, \ldots, 1)$. The difference $\mathbf{2} - I \cdot n$ is equal to M_n, and since $4n \neq n$, M_n has trivial kernel. The only coefficient on the leading term of P_Γ is $-\det M_n$, which is nonzero. □

Lemma 4. *There exists a nonzero polynomial vanishing over $\Sigma_\Gamma^{\text{sing}}$ whose leading homogeneous part has weight at most $n-2$.*

Proof. Denote by P'_Γ the first derivative of P_Γ with respect to z_1. By Lemma 3, P_Γ is a polynomial with terms of the form $z_{i_1}^2 \cdots z_{i_k}^2$. Hence, every z_1-term in $2 \cdot P_\Gamma$ is equal to a term of $z_i \cdot P'_\Gamma$. In particular, $f = 2 \cdot P_\Gamma - z_i \cdot P'_\Gamma$ contains no z_1-terms. Further, terms in the polynomial f appear like $z_{i_1}^2 \cdots z_{i_k}^2$ with $i_1 > 1$. If the weight of the leading homogeneous part of f is less than $n-2$, we are done. If not, then the leading homogeneous part of f is a nonzero constant times $z_2^2 \cdots z_n^2$. The derivative P'_Γ is divisible by z_1. Let $g = P'_\Gamma / z_1$. According to Lemma 3, its leading homogeneous term is

$$-2 \cdot \det M_n \cdot z_2^2 \cdots z_n^2$$

which is nonzero. Find a constant c such that $f - c \cdot g$ has a leading homogeneous part of weight less than $n-2$.

We claim that $f - c \cdot g$ is nonzero and vanishes over $\Sigma_\Gamma^{\text{sing}}$. Both P_Γ and P'_Γ vanish over Σ_Γ. Since Σ_Γ lies on the torus, no coordinate functions $z_1, \ldots, z_n$ can vanish over Σ_Γ. Rather, g vanishes over Σ_Γ. The sum $f - c \cdot g$ therefore vanishes over Σ_Γ. If this polynomial were zero, then one could write

$$2 \cdot P_\Gamma = z_1 \cdot P'_\Gamma + c \cdot g = g \cdot (z_1^2 + c).$$

This would suggest P_Γ is reducible. For star graphs Γ with three or more edges, this cannot occur. Instead, $f - c \cdot g$ is nonzero. $\square$

Proof of Theorem 1. Let $H \subset \mathbb{P}^n$ denote the hyperplane at infinity, i.e. $H = Z(z_0)$. Allow A to denote the projective closure of $\Sigma_\Gamma^{\text{sing}}$. Consider the intersection of A with H. All the partials of P_Γ vanish over $\Sigma_\Gamma^{\text{sing}}$ by definition. The leading homogeneous part of the i-th partial takes the form:

$$m_i = -2 \cdot \det M_n \cdot z_1^2 \cdots z_{i-1}^2 \cdot z_{i+1}^2 \cdots z_n^2,$$

according to Lemma 3. All other terms, besides m_i, in ${}^{\text{proj}}P_\Gamma$, are divisible by z_0. Thus, for $\beta(\partial P_\Gamma / \partial z_i)$ to vanish on $A \cap H$, the only nonzero term m_i, after substituting $z_0 = 0$, must vanish. The projective closure $\overline{V_\Gamma^{\text{sing}}}$ is contained the vanishing locus of all the $\beta(\partial P_\Gamma / \partial z_i)$, so

$$A \cap H \subset \overline{V_\Gamma^{\text{sing}}} \cap H \subset Z(m_1, \ldots, m_n, z_0).$$

For all monomials $m_1, \ldots, m_n$ to vanish, at least two of $z_1, \ldots, z_n$ must be zero. Thus,

$$A \cap H \subset \bigcup_{1 \le i < j \le n} H_{i,j} = V,$$

where $H_{i,j}$ is the projective plane $Z(z_0, z_i, z_j)$. Pick an irreducible component W of $A \cap H$, which represents its dimension, i.e. $\dim W = \dim A \cap H$. The irreducible components of V are percisely projective planes of the form $H_{i,j}$, meaning $W \subset H_{i,j}$ for some i, j.

We claim that the inclusion $W \subset H_{i,j}$ is strict. According to Lemmas 2 & 4, we may find a polynomial, vanishing over $\Sigma_\Gamma^{\text{sing}}$, whose leading homogeneous part contains a term missing both the variables z_i and z_j. The homogenization of this polynomial is necessarily nonzero after substituting $z_0 = z_i = z_j = 0$. Let p be the nonzero polynomial obtained after homogenization and substitution. With A being the projective closure of $\Sigma_\Gamma^{\text{sing}}$, $W \subset Z(p) \cap H_{i,j}$. Since p is a polynomial in all variables except z_0, z_i, z_j, p cannot lie in $I(H)$. That is, $Z(p) \cap H_{i,j}$, and therefore W, is a proper subset of $H_{i,j}$.

The dimension of W is at most $n - 4$ since it is a proper subset of $H_{i,j}$, which has codimension 3 in $\mathbb{P}^n$. Recall that $\dim W = \dim A \cap H$. By Prop. 2, one has

$$\dim A \le \dim A \cap H + 1 = \dim W + 1 \le n - 3.$$

The projective closure of $\Sigma_\Gamma^{\text{sing}}$ and its usual affine closure have the same dimension. Prop. 1 gives the bound $\dim \Sigma_\Gamma^{\text{sing}} \le n - 3$. Since the dimension of the secular manifold is exactly $n - 1$, the star graph Γ has property $(*)$, as claimed $\square$.

5. Future Directions

Tree Graphs. Tree graphs appear to share similar properties with star graphs. In particular, it seems Lemma 3 holds. The terms in the secular determinant all the take the form $z_{i_1}^2 \cdots z_{i_k}^2$. The key fact about star graphs is the symmetry of P_Γ under permutation. This is no longer true in general for tree graphs, so intersecting with the hyperplane at infinity does not immediately produce the desired result. This hyperplane was introduced mainly for its simplicity. The same method could apply to tree graphs with a different hyperplane or even a hypersurface.

Shrinking Edges. Given a graph Γ, we can consider quantum graphs on Γ where some edge length is zero. Let Γ' be the graph obtained from Γ by shrinking the edge, whose length is set to zero. Assigning Γ' the same edge lengths as Γ for all remaining edges, the spectrums of the two quantum graphs agree, assuming the shrunken edge was not a loop. Setting an edge length to zero is equivalent to assigning its associated variable the value 1, and the secular determinant $P_{\Gamma'}$ is obtained from P_Γ by substitution. The secular manifolds are also related. If Γ has n edges, we may embed $\Sigma_{\Gamma'}$ in the affine hyperplane $H = Z(z-1)$, where $\Sigma_\Gamma \cap H = \Sigma_{\Gamma'}$.

The relationship between $\Sigma_\Gamma^{\text{sing}}$ and $\Sigma_{\Gamma'}^{\text{sing}}$ doesn't appear as obvious, although it is certainly true that $\Sigma_\Gamma^{\text{sing}} \cap H \subset \Sigma_{\Gamma'}^{\text{sing}}$. For basic examples, equality seems to hold. Understanding this relationship might be important. We ask the following question: *If Γ' satisfies property $(*)$, does Γ?* Ideally, one would apply intersection theory to obtain a relationship between the dimensions of $\Sigma_\Gamma^{\text{sing}}$ and $\Sigma_{\Gamma'}^{\text{sing}}$. This requires one of two things: relating either (1) the irreducible components or (2) the projective closures of $\Sigma_\Gamma^{\text{sing}}$ and $\Sigma_{\Gamma'}^{\text{sing}}$. If this is successful, induction might be applicable, reducing the codimension 2 conjecture to graphs which are one shrinking operation away from possessing a loop.

Graph Surgery. As evident in the factorization theorem for secular determinants, loops cause substantial issues. Less obvious is the fact that the Betti number of a graph Γ appears to effect the secular determinant. The property of tree graphs, as outlined above, no longer holds. In particular, given an non-looped edge with variable z, there might be a sequence of shrinking operations sending this edge to a loop. If this so, then then P_Γ has a degree 1 term in z. As the Betti number increases, the terms of P_Γ might become more unpredictable.

Shrinking non-looped edges does not alter the Betti number. Instead, a different surgery operation should be introduced. We suggest considering either (1) edge deletion or (2) shrinking looped-edges. The second operation appears harder because it requires us to shrink to a graph with a loop before removing it. Instead, edge deletion changes the Betti number of the graph without introducing loops. This is ideal considering the codimension 2 conjecture requires our graphs to be loop-free. In particular, for an edge deletion operation sending Γ to Γ', we ask the following questions: *How are the secular determinants P_Γ and $P_{\Gamma'}$ related? How are the singular sets of Σ_Γ and $\Sigma_{\Gamma'}$ related? If Γ' has property $(*)$, does Γ?*

Appendix: Proof of Propositions 1 & 2

Proposition 1. *Let V be an algebraic set in $\mathbb{C}^n$, and let M denote the intersection $V \cap \mathbb{T}^n$. The regular points of M form a manifold whose dimension is at most* $\dim V$.

Proof. Every affine algebraic set V in $\mathbb{C}^n$ is a finite union of its irreducible components. A point of V is *regular* if it is contained in percisely one irreducible component and further a regular point of that variety. Proving the claim for varieties thus proves the claim for algebraic sets; assume V is a variety.

Pick some generators $f_1, \ldots, f_m$ from the vanishing ideal of V, so that V is the zero level set of mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ given by $P \mapsto (f_1(P), \ldots, f_m(P))$. Further define G as the map sending a point $(z_1, \ldots, z_n)$ to

$$(|z_1|^2 - 1, \ldots, |z_n|^2 - 1).$$

The zero level set of G is the n-torus $\mathbb{T}^n$. This gives a smooth map

$$(F, G) : \mathbb{C}^n \to \mathbb{C}^m \times \mathbb{R}^n$$

whose zero level set is percisely $V \cap \mathbb{T}^n$. Regular points of V correspond to elements z where $T_z V$ is percisely the kernel of dF_z. The map (F, G) is regular in the intersection $V^{\text{reg}} \cap \mathbb{T}^n$, and it suffices to bound the dimension of $T_z(V \cap \mathbb{T}^n)$ at a regular point z.

The tangent space to $V \cap \mathbb{T}^n$ at a regular point is the intersection of $T_z V$ and $T_z \mathbb{T}^n$, where $T_z \mathbb{T}^n$ is a real vector space and $T_z V$ is a complex one. Consider the following equality:

$$T_z \mathbb{T}^n = i \cdot \operatorname{diag}(z)\big(\mathbb{R}^n\big),$$

where $\operatorname{diag}(z)$ is the diagonal matrix with entires $z_1, \ldots, z_n$. Since z is on the torus, the operator $i \cdot \operatorname{diag}(z)$ is invertible. Then

$$\dim(T_z V \cap T_z \mathbb{T}^n) = \dim(i \cdot \operatorname{diag}(-\overline{z})\big(V\big) \cap \mathbb{R}^n).$$

Because $i \cdot \operatorname{diag}(-\overline{z})$ is a complex linear transformation, its image of $T_z V$ is also a complex space of the same dimension as V. The intersection $i \cdot \operatorname{diag}(-\overline{z})\big(V\big) \cap \mathbb{R}^n$ is real. If we select a basis, then its 'complexification' in $T_z V$ is an independent set. This produces the inequality $\dim V^{\text{reg}} \cap \mathbb{T}^n \leq \dim V$. □

Proposition 2. *Let V be an algebraic set in $\mathbb{P}^n$, and H a hypersurface. If* $\dim V \geq 1$, *then*

$$\dim V \leq \dim V \cap H + 1.$$

Proof. If $A \subset \mathbb{P}^n$ has dimension 1, then we may find an irreducible component of A, call it V, such that $\dim A = \dim V$. It suffices to prove the result for a projective variety V. The intersection of a variety V with positive dimension and a hypersurface H is always nonempty (see Theorem 7.2, Chapter I of [5]). Now, pick an open affine subset U containing some point of their intersection. Then $V' = V \cap U$ is an affine variety of the same dimension as V, while $H' = H \cap U$ is an affine hyperplane intersecting V'. By Exercise 2.6, Chapter I of [5], the affine version of Prop. 2 holds. Thus,

$$\dim V = \dim V' \leq \dim V' \cap H' + 1 \leq \dim V \cap H + 1,$$

where the far left and right (in)equalities occur because dimension is preserved by dense open subsets. □

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References

[1] L. Alon, Generic Laplacian Eigenfunctions on Metric Graphs, Journal d'Analyse Mathématique, preprint, arXiv 2203.16111.

[2] G. Berkolaiko. "An Elementary Introduction to Quantum Graphs". Contemporary Mathematics, 700, AMS 2017, arXiv:1603.07356 [math-ph].

[3] Y. Colin de Verdière, Semi-classical measures on quantum graphs and the Gauß map of the determinant manifold, Annales Henri Poincaré, 16 (2015), pp. 347–364. also arXiv:1311.5449.

[4] L. Friedlander. "Genericity of simple eigenvalues for a metric graph". Israel J. Math. 146 (2005), pp. 149–156.

[5] R. Hartshorne, (2010). *Algebraic geometry*. Springer.

[6] P. Kurasov and P. Sarnak, "The additive structure of the spectrum of a Laplacian on a metric graph" (unpublished) (2020)