2023 INDIANA UNIVERSITY REU PROJECT DESCRIPTIONS

Please remember that we have a pre-REU preparation program, a intensive chance to get up to speed on any prerequisites you may be weaker on.

1. Louis Fan: Extinction probabilities and longtime behaviors of stochastic reaction networks

Stochastic reaction networks are powerful mathematical tools to understand stochastic extinction, spatial spread and other phenomena in bio-chemical systems. Mathematically, they are often formulated as Markov chains, ordinary differential equations or stochastic differential equations, depending on the relevant scaling regimes.

In these few weeks, we will

(i) (i) develop a new stochastic reaction network important for the study of intracellular viral kinetics;
(ii) (ii) analyze its mathematical properties including extinction probability and asymptotic behaviors (long-time, small noise);
(iii) (iii) perform stochastic simulations to contrast with the deterministic modeling approach;
(iv) and (iv) distill the biological insight enabled by the mathematical and computational results.

Prerequisites. Basic knowledge in probability and combinatorics, including the concepts of random variables, joint distribution, conditional probability and central limit theorem.

Familiarity with a computer program (such as Matlab, Python, R) would be a plus, but not required.

2. Chris Judge: The topography of quantum states in polygons

Imagine a single quantum particle confined to a 2-dimensional polygon. It moves around the polygon bouncing off sides when it meets them. Classical quantum theory does not predict the exact location or momentum of the particle. Instead it provides, for example, the likelihood that the particle will lie in any given part of the polygon. Mathematically, this likelihood is the integral of a function on the polygon, the 'wave function'.

The purpose of this project is to search for wave functions whose level sets have interesting topological features. For example, is there a wave function whose level set is a figure eight? We will explore such questions using both thought experiments and computer experiments. In the end, we will rigorously justify our observations with proofs.

Prerequisites. The student should be fluent in undergraduate linear algebra and undergraduate analysis. No prior experience with quantum physics is necessary though it might be helpful. Familiarity with software such as Mathematica, Maple, Matlab, or Sage is preferable.

3. Matvei Libine: Conformal Transfomations on $\mathbb{R}^{p,q}$ and Clifford Valued Functions

This project is motivated by complex analysis, but a course in complex variables is *not* a prerequisite. Complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ are special cases of Clifford algebras, and many results of complex analysis have analogues in quaternionic analysis as well as Clifford analysis. The goal of this project is to study symmetry properties of Clifford algebra analogues of differentiable functions—called *monogenic functions*—under conformal or Möbius transformations.

Here is an example typically covered in a complex variables course. For each invertible matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in \mathbb{C}$, one can define a *fractional linear transformation* or *Möbius transformation* on $\mathbb{C} \cup \{\infty\}$ sending

$$z \mapsto \frac{az+b}{cz+d}.$$

If $f(z)$ is a complex-differentiable (or *holomorphic*) function, then so is $f\big((az+b)(cz+d)^{-1}\big)$.

A higher dimensional analogue of this setup involves a generalized Minkowski space $\mathbb{R}^{p,q}$ with $p, q = 0, 1, 2, \dots$. This is the Euclidean vector space $\mathbb{R}^{p+q}$ with indefinite quadratic form

$$Q_{p,q}(\overrightarrow{v}) = (x_1)^2 + \cdots + (x_p)^2 - (x_{p+1})^2 - \cdots - (x_{p+q})^2.$$

We denote by $O(p,q)$ the set of $(p+q) \times (p+q)$ real matrices preserving this quadratic form, i.e. $O(p,q)$ consists of all real matrices A such that $Q_{p,q}(A\overrightarrow{v}) = Q_{p,q}(\overrightarrow{v})$.

For each $(p+q+2) \times (p+q+2)$ matrix in $O(p+1, q+1)$, there is a way to produce a transformation on $\mathbb{R}^{p,q}$ which is *conformal*. (Conformal transformations generalize metric-preserving transformations in differential geometry.)

A monogenic function is defined on (a subset of) $\mathbb{R}^{p,q}$, takes values in an appropriate Clifford algebra and satisfies certain differential conditions. If f is such a monogenic function and φ is a conformal transformation on $\mathbb{R}^{p,q}$, then typically the composition $f \circ \varphi$ is *not* monogenic. However, it is well-known that, when $q = 0$, a conformal transformation φ of $\mathbb{R}^p$ can be expressed as

$$\overrightarrow{v} \mapsto (a\overrightarrow{v} + b)(c\overrightarrow{v} + d)^{-1}$$

and there is a certain factor that makes this composition monogenic:

$$\frac{\overline{c\overrightarrow{v} + d}}{\|c\overrightarrow{v} + d\|^p} \cdot f\big((a\overrightarrow{v} + b)(c\overrightarrow{v} + d)^{-1}\big) \quad \text{is monogenic.}$$

The goal of this project is to find analogues of this formula when $q \neq 0$.

Prerequisites. This project is motivated by complex analysis, but a course in complex variables is *not* a prerequisite.

Strong background in multivariable calculus, linear algebra (including quadratic forms), and a course on algebra involving rings and modules (needed to understand the definitions of Clifford algebras and modules).

Exposure to an introductory course on complex variables would help to understand the motivation behind the project, but is not a prerequisite.

Be aware that the project involves a non-commutative setting which may take some time to get used to.

4. Larry Moss: Natural Logic

Larry Moss has been investigating systems of logic where one does formal reasoning, but instead of using a standard system of logic, one reasons in natural language or something close to it. This big project has a lot of spinoffs that could be REU projects. Some border on combinatorics, some involve programming and/or theoretical computer science, and some are closer to linguistics.

Prerequisites. A good understanding of completeness theorems from a class on logic. Additionally, it would be nice to have some background in one of the extra topics listed above.

5. Josh Southerland: Spectra of graphs

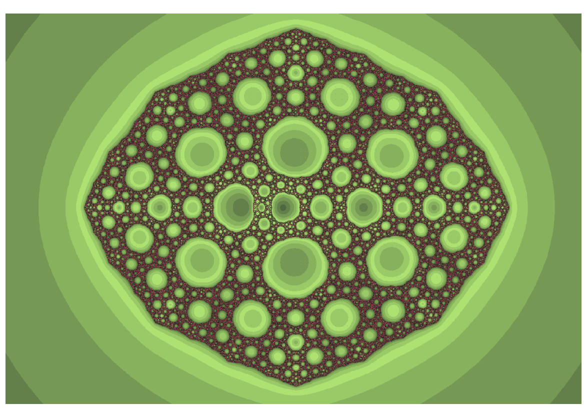
Graphs embedded on surfaces are famously used to prod at the properties of operators on the surface, such as the Laplacian. In the 1970s, work of Dodziuk and Patodi showed how one can use a discretized operator (for example, a combinatorial Laplacian) to understand properties of the corresponding operator. Müller famously built upon these ideas to prove a conjecture of Ray and Singer.

In the first week of the program, we will review (or learn) elementary graph theory, and introduce the notion of the adjacency matrix and the combinatorial Laplacian. Our first goal will be to explore embedded 4-regular graphs on tori and develop a concrete understanding of how the combinatorial Laplacian converges to a Laplacian on the surface. Then, generalizing from tori, we will explore graphs embedded on square-tiled surfaces: a type of flat surface with conical singularities. This line of inquiry may involve elements of differential topology.

Prerequisites. Linear algebra, multivariable calculus, and topology (or an introduction to general topology as a part of an analysis course). It would also be desirable to also have some programming experience and knowledge of differential geometry.

6. Dylan Thurston: Expansivity of fractals

In Summer 2022, we started an investigation of how strongly expanding various natural functions are, and found some surprisingly interesting phenomena. More precisely, we looked at rational functions of a complex variable and looked at how they acted on the Julia set. There were many unanswered questions; for instance, of the 17 different functions we looked at, there was one, $f(z) = (4/27)(z-1)^3/z$, with the Julia set shown below, for which we were unable to conjecture the expansion rate. We will see if we can resolve the expansion for this case, and push further to wider classes of examples to see if we can establish a more general theory.



Prerequisites. Complex analysis, basic topology.

7. Shouhong Wang: Pattern formation and dynamic transitions of the 1D Cahn-Hilliard equation

In this project, we explore the high-order central manifold reductions of the one dimensional Cahn-Hilliard equation with or without nonlocal terms, and study its dynamic behavior near the first critical threshold.

Prerequisites. Multivariable calculus, basic Ordinary Differential Equations (ODE) and Partial Differential Equations (PDE)

8. Kevin Zumbrun: Nonconvex optimization and asynchronous coalition in 3-player games

The difficulties in analyzing 3- and more-player games are well-known, introduced by possibility of collussion/coalition between different subsets of players. Here, we propose an aspect that does not seem to have been studied, namely, optimization of coalition strategies without communication or time-synchronization within the play of the game. That is, the game would be played repeatedly, with colluding players isolated, and not informed which iteration of the game they are playing at a given time- hence the title "asynchronous" coalition play. This entails, for an associated trilinear payoff function $\phi(x, y, z)$ where each of x, y, z are n-dimensional probability vectors, the minimization problem

$$\min_{y,z}\Big(\max_{x} \phi(x, y, z)\Big),$$

a nonconvex minimization problem defined implicitly by the maximum with respect to x. The goal is to develop efficient and mathematically well-supported numerical methods to approximate the solution of this problem. Nonconvex minimization is the object of a great deal of current attention, pertaining to machine learning, AI, and any number of related issues, and this project is an excellent chance for a student to learn the basics. But, also, the implicit definition of the objective function is a new wrinkle specific to games, that should lead to interesting new issues in the algorithm development.

Note: This project will have two students working as a team.

Prerequisites. Programming experience (Python), linear algebra, and vector calculus needed. Some familiarity with mathematical proof (a "bridge" type analysis or algebra course) helpful.