Low-dimensional musical pitch and chord spaces

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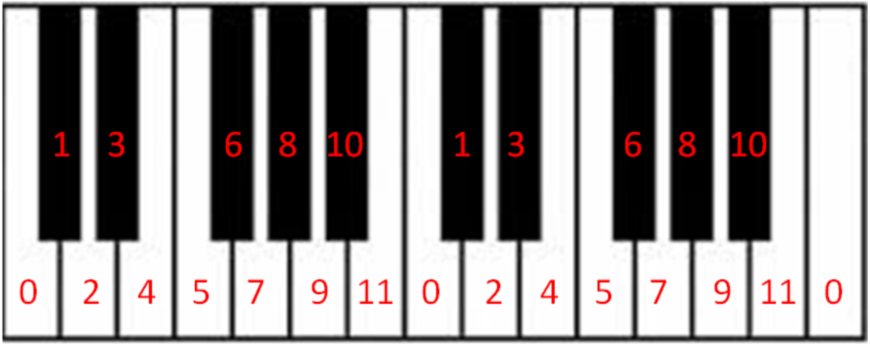
Outline

Definitions

- Frequency
- Pitch
- Chroma
- Octave
- Pitch class
- *n*-chord

1-dimensional pitch space: a pitch-class model [$\mathbb{Z}_{12}$]



Aural motivation

- Functional perception of pitch
- Understanding tonal / atonal music

Measuring distance between pitches: d_1

Definition

Let x, y be pitch classes. Then the function $d_1 : \mathbb{Z}_{12} \times \mathbb{Z}_{12} \to \{0, \ldots, 6\}$ given by $d_1(x, y) = \min(x - y \bmod 12, y - x \bmod 12)$ gives their *pitch-class distance*.

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Example

The pitch-class distance between $x = 11$ and $y = 2$ is $\min(2 - 11 \bmod 12, 11 - 2 \bmod 12) = \min(-9 \bmod 12\ , 9 \bmod 12\ ) = \min(3, 9) = 3$.

n-chord spaces

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The n-chord space is the nth Cartesian product of 1-dimensional pitch space mod permutations of its ordinates, ie $(\mathbb{Z}_{12})^n / \sum_{\mathsf{n}}$.

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Example

Equivalency classes in 3-chord space

Measuring distance between chords: d_n

Definition

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be n-chords. Then the function $d_n : (\mathbb{Z}_{12})^n / \sum_{\mathrm{n}} \times (\mathbb{Z}_{12})^n / \sum_{\mathrm{n}} \to \{0, \ldots, 6n\}$ given by $d_n(x, y) = \min_{\sigma \in S_n} \sum_i d_1(x_i, y_{\sigma(i)})$ gives their *chord-class distance*.

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d_n is a metric on $(\mathbb{Z}_{12})^n/\sum_n$ for $2 \leq n \leq 4$.

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Conjecture

d_n is a metric on $(\mathbb{Z}_{12})^n/\sum_n$ for $n \geq 5$.

Measuring distance between chords: d_n

Example

Let $x = (0, 3)$ and $y = (5, 7)$.
Then $d_2(x, y) = \min(d_1(0, 5) + d_1(3, 7), d_1(0, 7) + d_1(3, 5)) =$
$\min(5 + 4, 5 + 2) = \min(9, 7) = 7$.

Benefits and limitations of d_n

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- Benefits
  - Measuring dissonance
  - Understanding atonal music
  - Works for arbitrarily large n-chords

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- Benefits
  - Measuring dissonance
  - Understanding atonal music
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- Limitations
  - Coarseness
  - Meaningless distances between degenerate n-chords.

Building a tonal dictionary

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Definition

For any specific permutation representation of an n-chord $(x_1, \ldots, x_n)$ with $n \geq 2$, its *interval vector* is the ordered $(n-1)$-tuple $\langle x_2 - x_1, \ldots, x_n - x_{n-1} \rangle$.

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Example

The interval vector of $(0, 3, 8)$ is $\langle 3, 5 \rangle$, while the interval vector of $(8, 0, 3)$ is $\langle 4, 3 \rangle$. In general, n-chords in the same equivalence class do not usually all have the same interval vector.

Building a tonal dictionary (cont.)

Example

Dictionary of 3-chords [triads]: a subset of $(\mathbb{Z}_{12})^3/\sum_3$

- Major: $\langle 4, 3 \rangle$
- Minor: $\langle 3, 4 \rangle$
- Diminished: $\langle 3, 3 \rangle$
- Augmented: $\langle 4, 4 \rangle$

Building a tonal dictionary (cont.)

Example

Dictionary of 4-chords [seventh chords]: a subset of $(\mathbb{Z}_{12})^4/\sum_4$

- $\langle 4, 3, 3 \rangle$
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Toward a tonal metric: the root of a 3- or 4-chord

Definition

The *root* $\tilde{x}$ of a 3- or 4-chord x in the dictionary is any pitch class in the chord for which there exists a permutation σ such that $\sigma(x_1) = \tilde{x}$ and $(\tilde{x}, \ldots, x_{\sigma(n)})$ has interval vector consisting solely of 3s and/or 4s.

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Example

The root of $x = (0, 3, 8)$ is 8 because $(8, 0, 3)$ has interval vector $\langle 4, 3 \rangle$.

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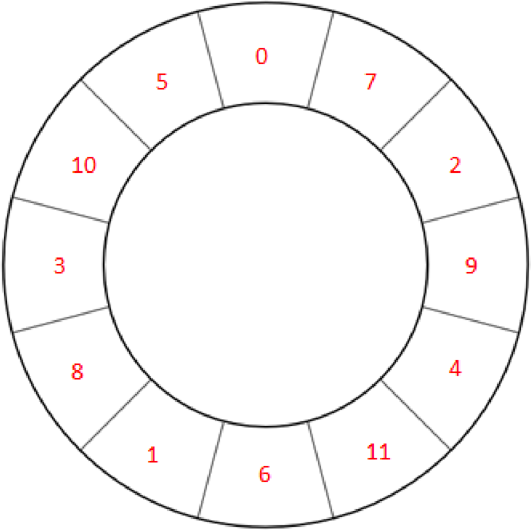
Example

The root of $x = (0, 3, 8)$ is 8 because $(8, 0, 3)$ has interval vector $\langle 4, 3 \rangle$.

Example

The root of $y = (3, 7, 11)$ is 3, 7, or 11 since y has interval vector $\langle 4, 4 \rangle$ so any permutation of y's ordinates has interval vector $\langle 4, 4 \rangle$.

Toward a tonal metric: the circle of fifths



Circle-of-fifths distance [for 3- or 4-chords in the dictionary]

Definition

The *circle-of-fifths distance* d_C between two 3- or 4-chords x and y in the dictionary with roots $\tilde{x} \equiv 7j$ and $\tilde{y} \equiv 7k$ is given by $d_C(x, y) = d_1(j, k)$. If at least one of the chords does not have a unique root, then d_C is understood to be the minimum over all possible roots of the chord(s).

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Example

The *Star Wars* Theme: $x = (7, 11, 2)$, $y = (4, 8, 11)$, $z = (10, 2, 5)$ so $\tilde{x} = 7$, $\tilde{y} = 4$ and $\tilde{z} = 10$, all with interval vector $\langle 4, 3 \rangle$.
Now $\tilde{x} \equiv 7(1)$ mod 12, $\tilde{y} \equiv 7(4)$ mod 12 and $\tilde{z} \equiv 7(10)$ mod 12 so
$d_C(x, y) = d_1(1, 4) = \min(-3, 3) = 3$ and
$d_C(y, z) = d_1(4, 10) = \min(-6, 6) = 6$.

Embedding 2-chord space in a Möbius Strip

Geometric intuition

$(S^1 \times S^1)/\sum_2$

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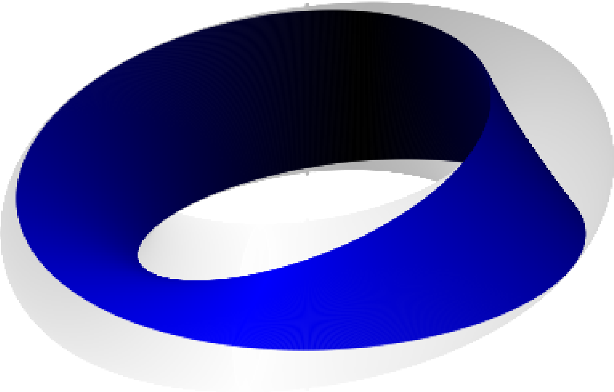

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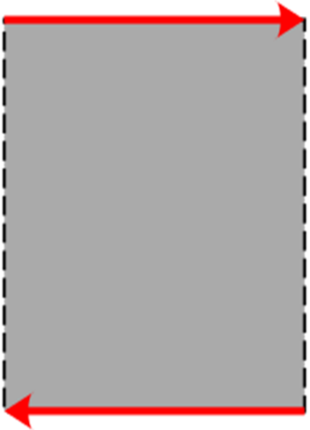

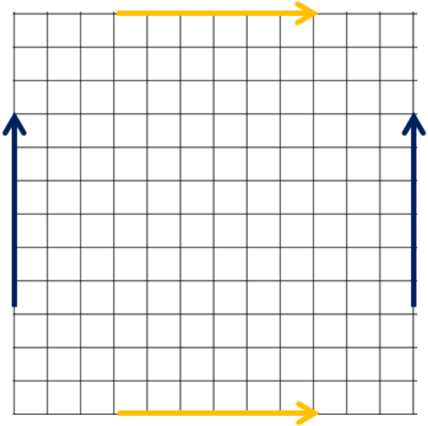

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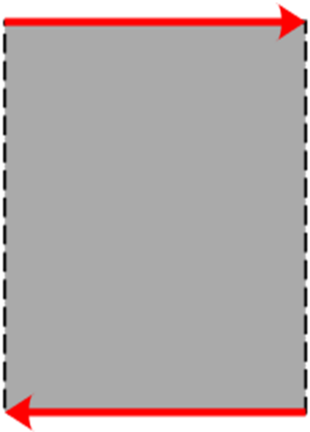

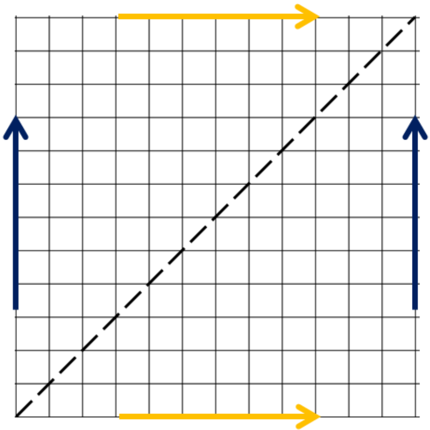

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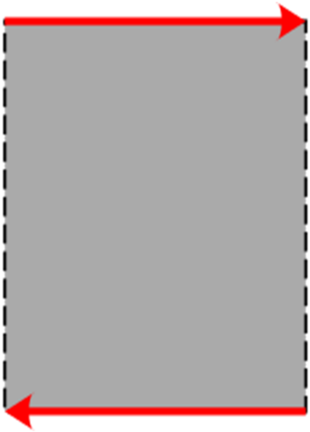

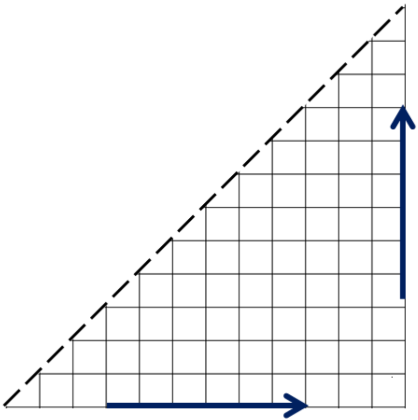

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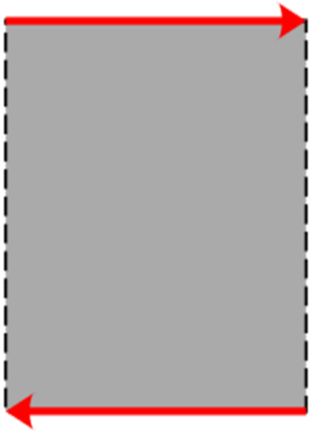

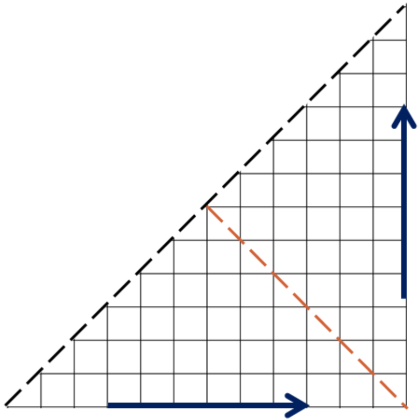

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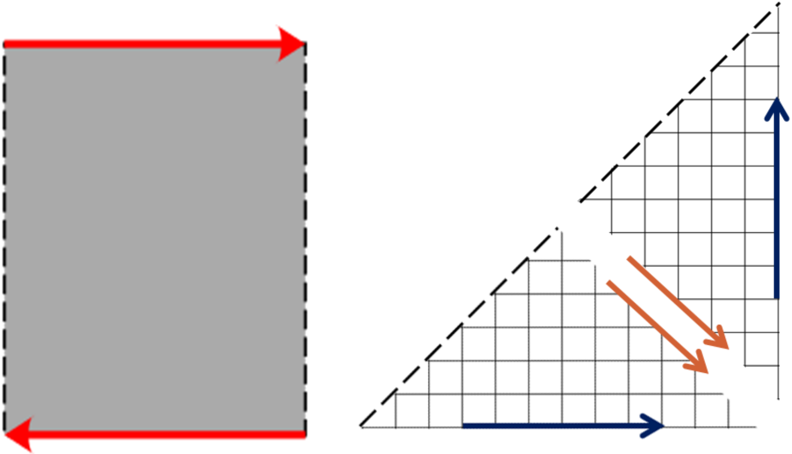

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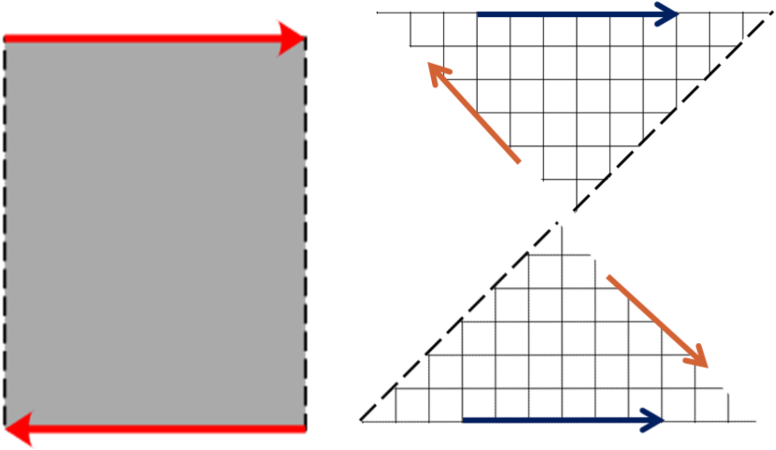

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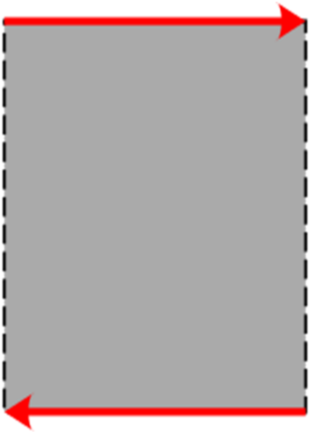

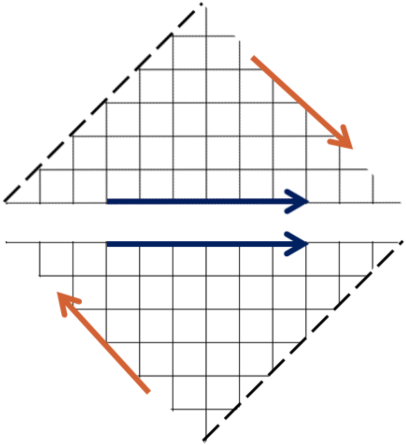

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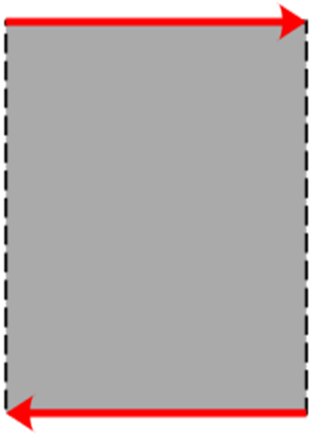

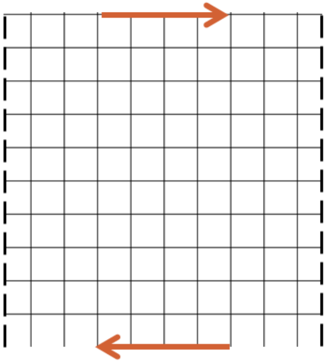

What's the boundary?

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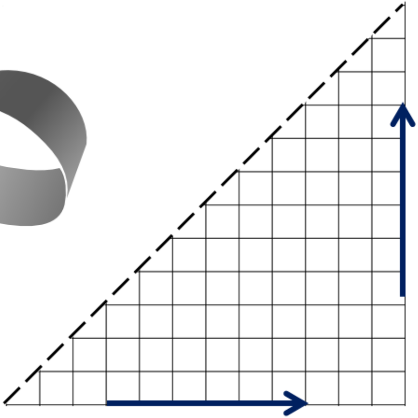

Embedding 3-chord space in a twisted triangular torus

Further applications

- Voice-leading
- Modulation in 19th century harmony
- Comparing Western and Eastern music