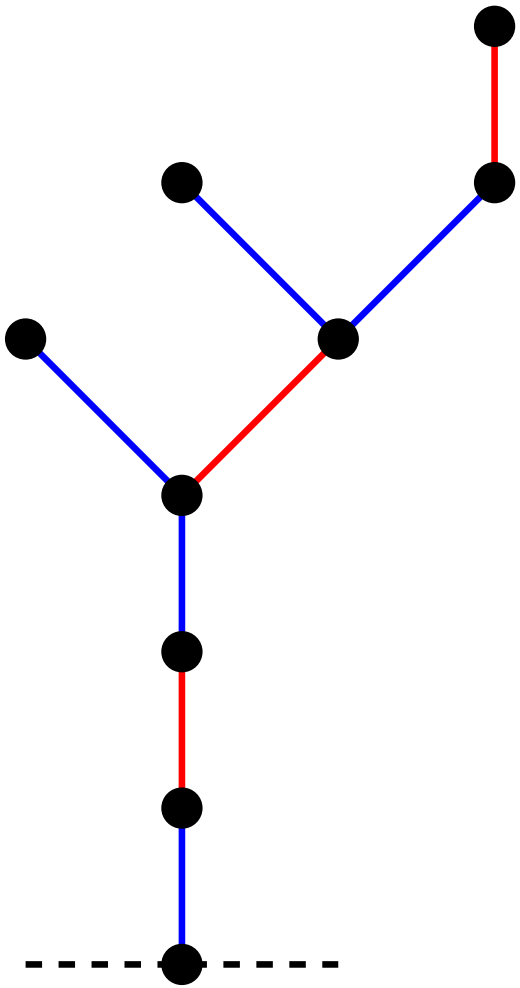
Playing Games with Numbers

Chase Abram

December 7, 2015

Hackenbush



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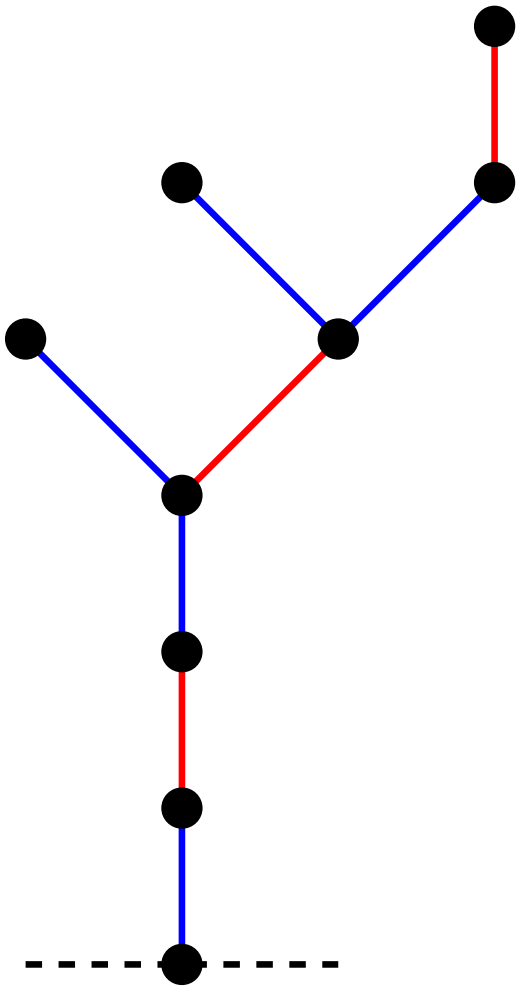

How to Play

1. Left Cuts Blue, Right Cuts Red
2. Unconnected branches disappear
3. First player unable to move loses

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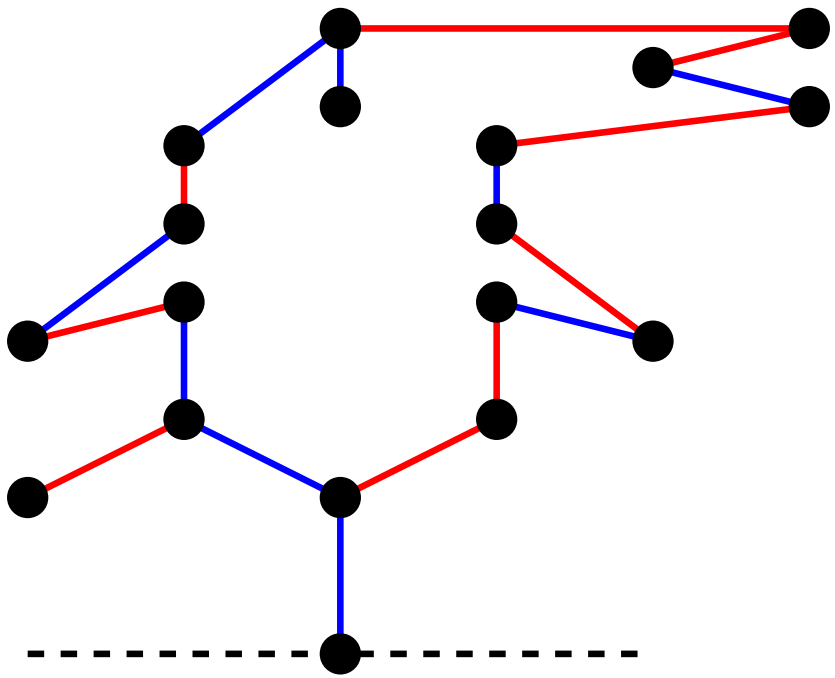

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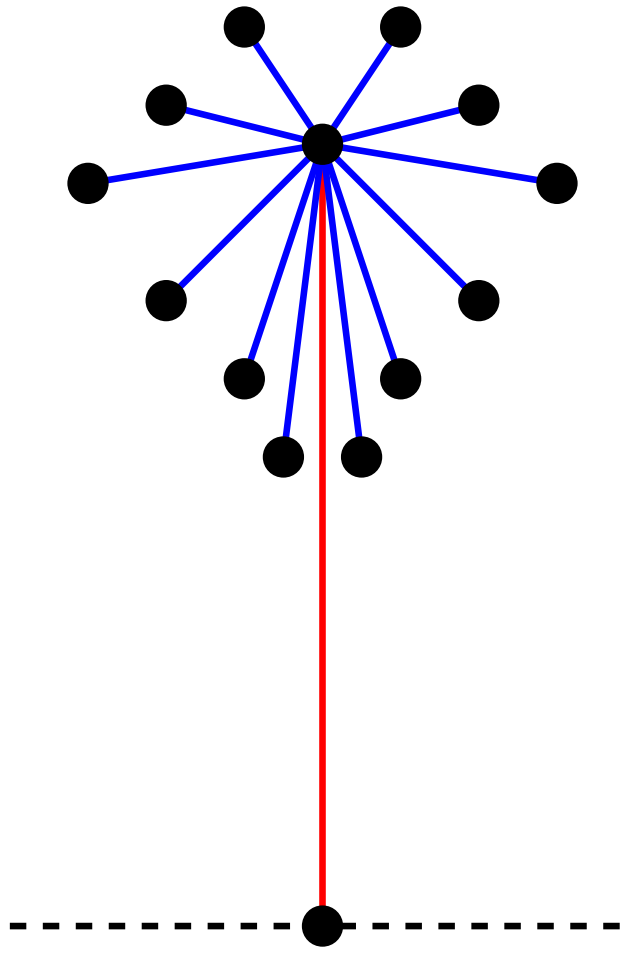

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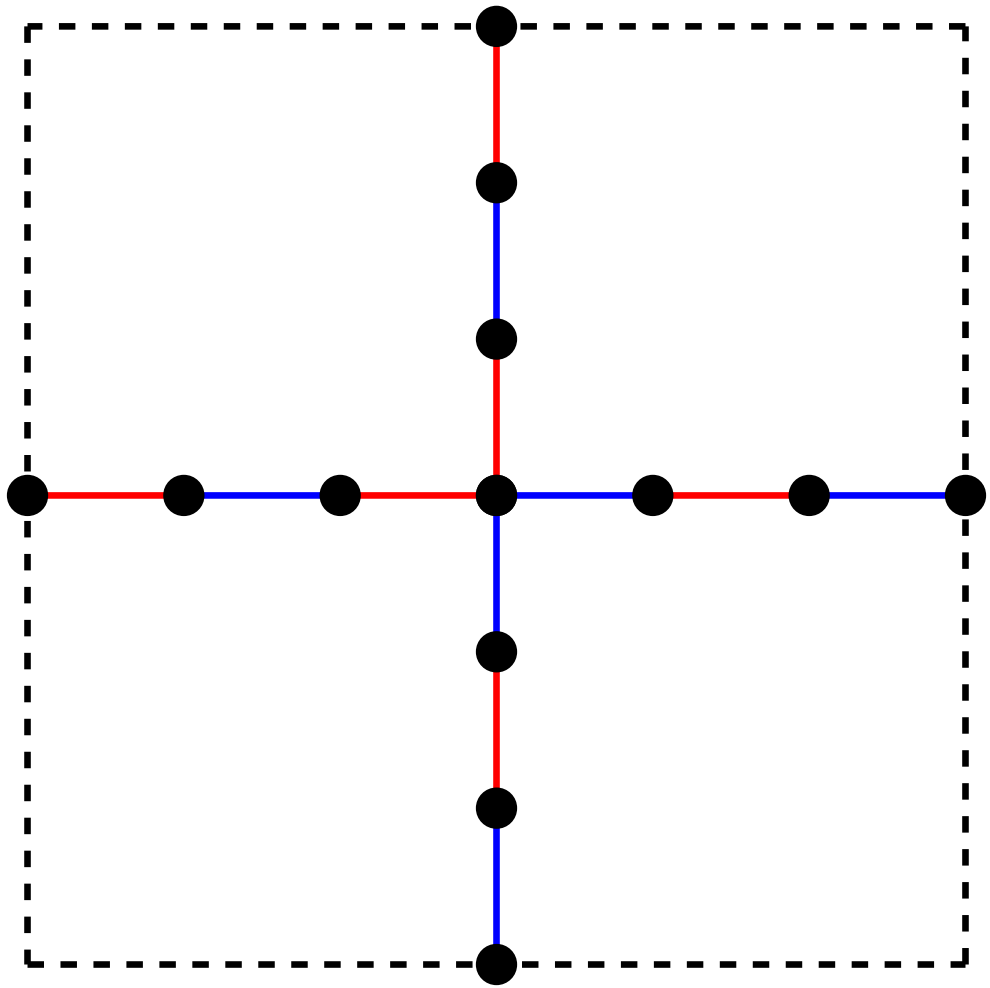

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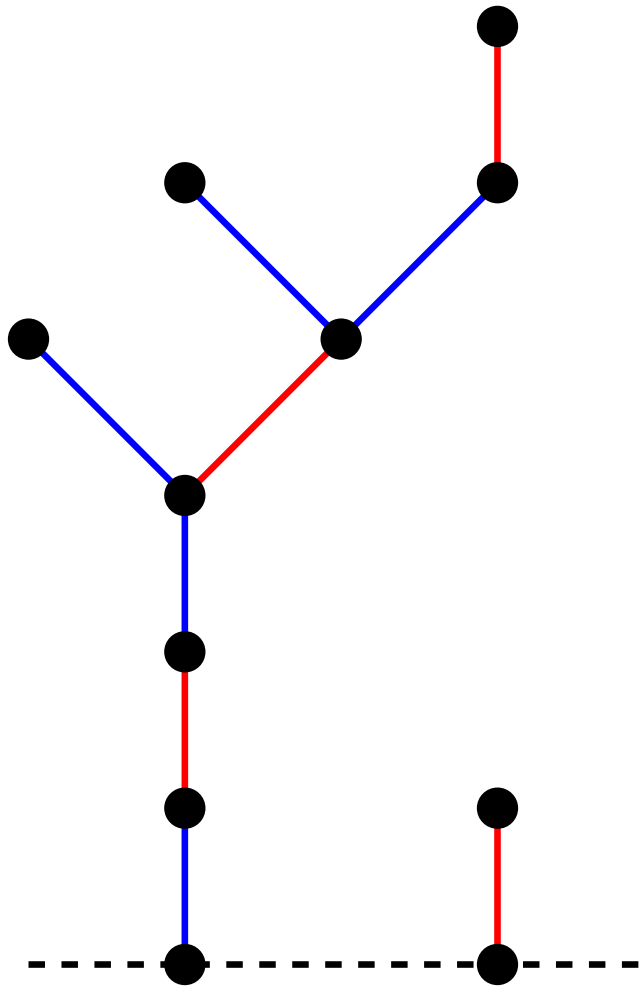

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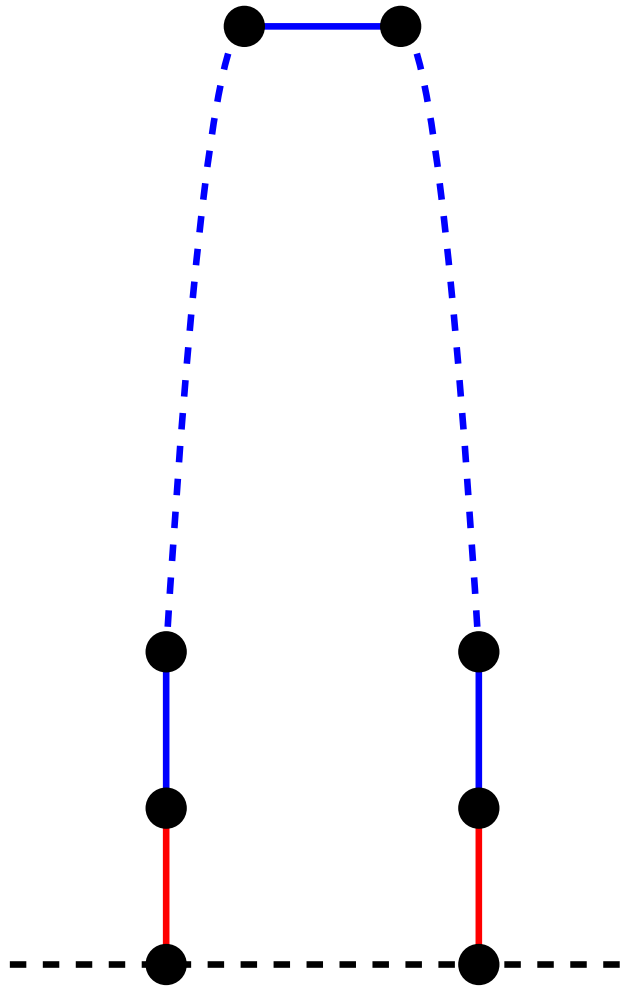

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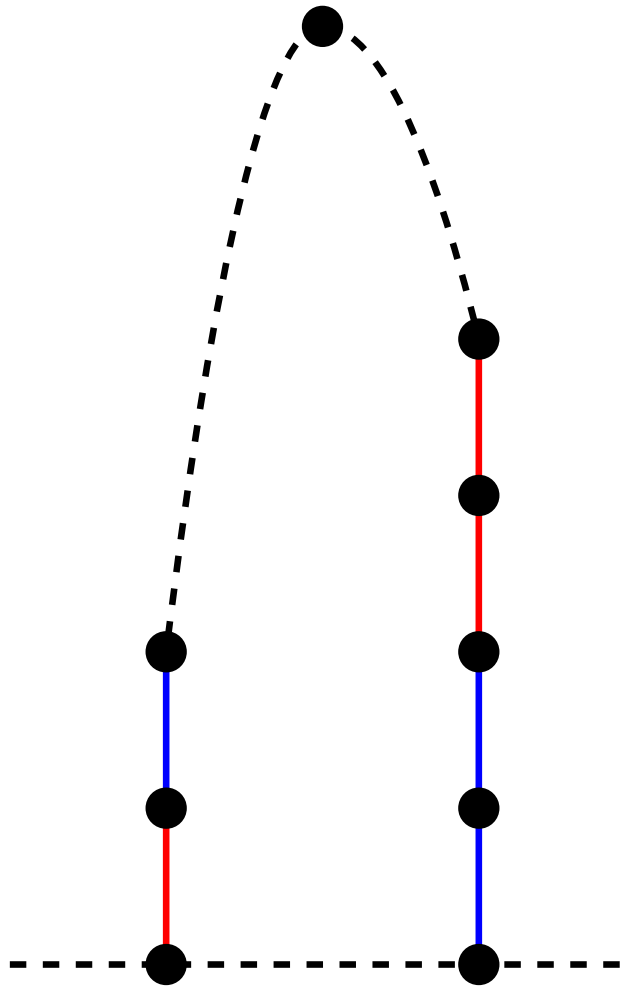

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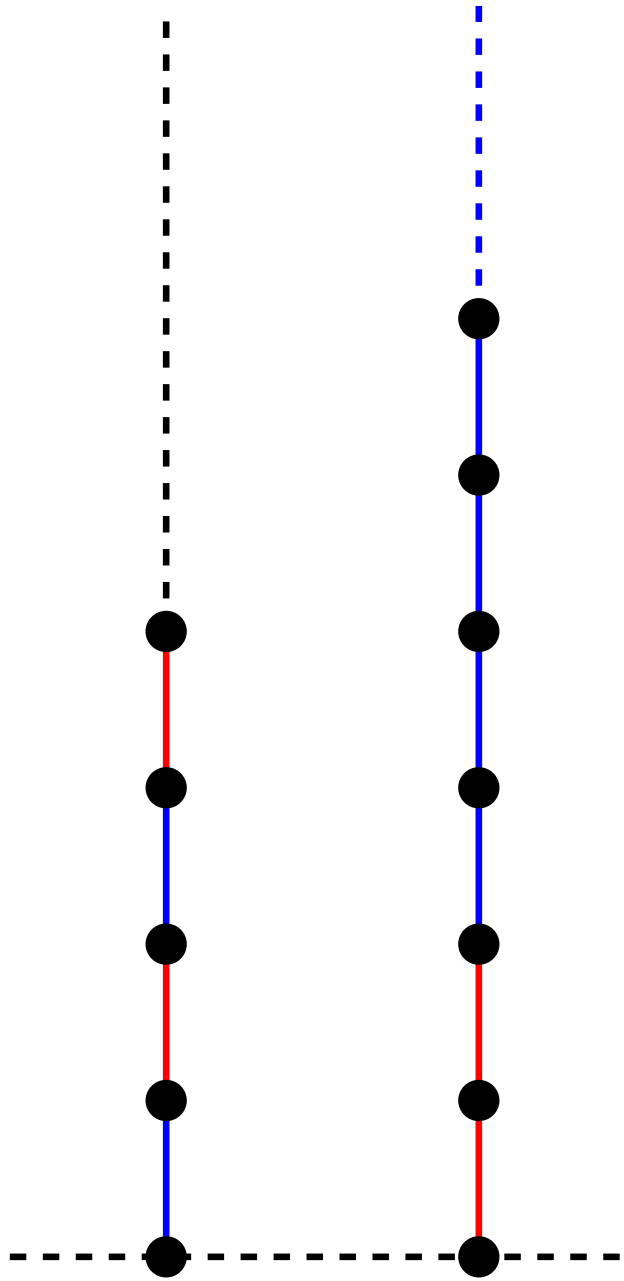

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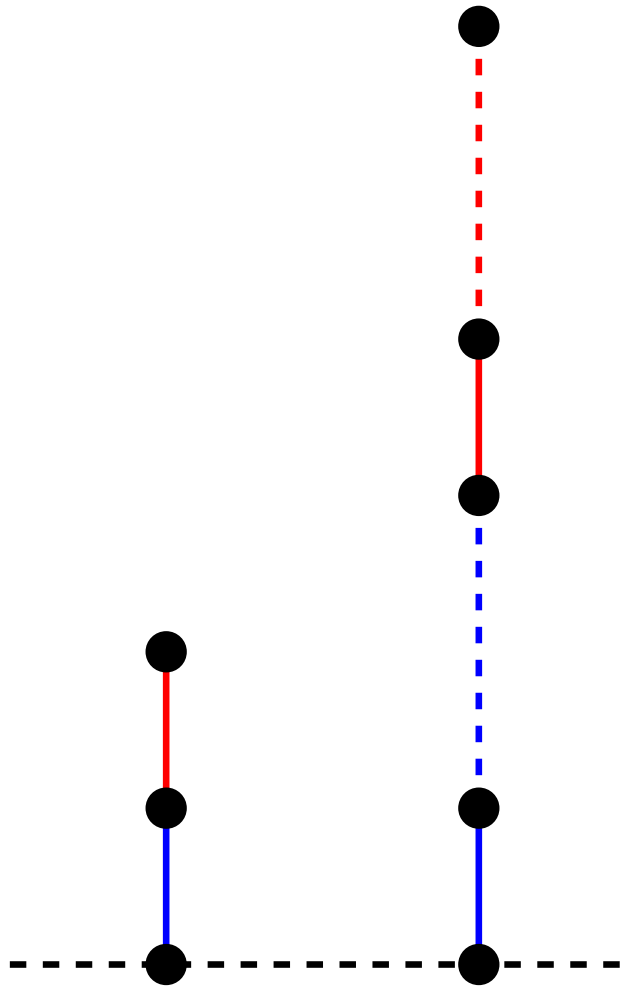


Surreal Numbers

$$0, \epsilon, 1, 2, e, \pi, \omega$$

All surreal numbers can be built from sets:

$$x = \{x^L | x^R\}$$

Every element of x^L must be less than every element of x^R.

Let's Build Numbers

Zero:

$$\{|\} = 0$$

Let's Build Numbers

Zero is used to build...

One:

$$\{0|\} = 1$$

And negative one:

$$\{|0\} = -1$$

Let's Build Numbers

One is used to build...

Two:

$$\{0, 1|\} = 2$$

And negative two:

$$\{|0, -1\} = -2$$

Let's Build Numbers

Two is used to build...

Three:

$$\{0, 1, 2|\} = 3$$

And negative three:

$$\{|0, -1, -2\} = -3$$

What About Those Surreal Numbers?

Consider

$$\{1, 2, 3, ..., n|\} = n + 1$$

What About Those Surreal Numbers?

Define

$$\{1, 2, 3, ...|\} = \omega$$

Let's Build Numbers

These are used to build:

$$\{0|1\} = \frac{1}{2}$$

and

$$\{-1|0\} = -\frac{1}{2}$$

Let's Build Numbers

Which are used to build:

$$\{0|\frac{1}{2}\} = \frac{1}{4}$$

and

$$\{-\frac{1}{2}|0\} = -\frac{1}{4}$$

Let's Build Numbers

Which are used to build:

$$\{\frac{1}{2}|1\} = \frac{3}{4}$$

and

$$\{-1|-\frac{1}{2}\} = -\frac{3}{4}$$

Let's Build Numbers

Also:

$$\{0|\frac{1}{4}\} = \frac{1}{8}$$

and

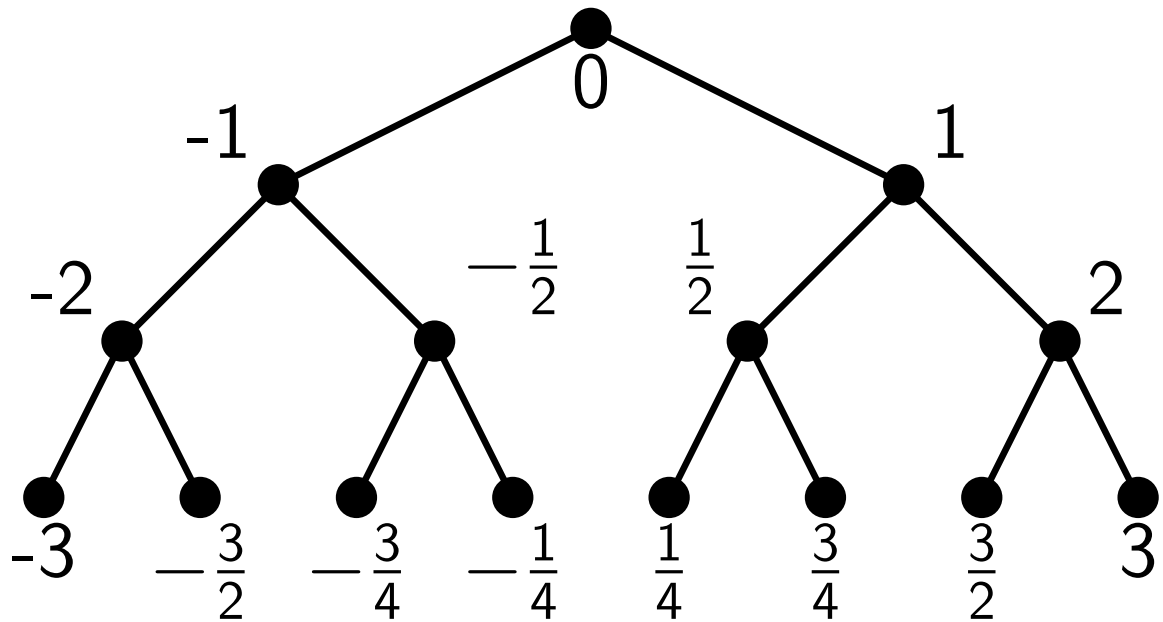
$$\{-\frac{1}{4}|0\} = -\frac{1}{8}$$

What About Those Surreal Numbers?

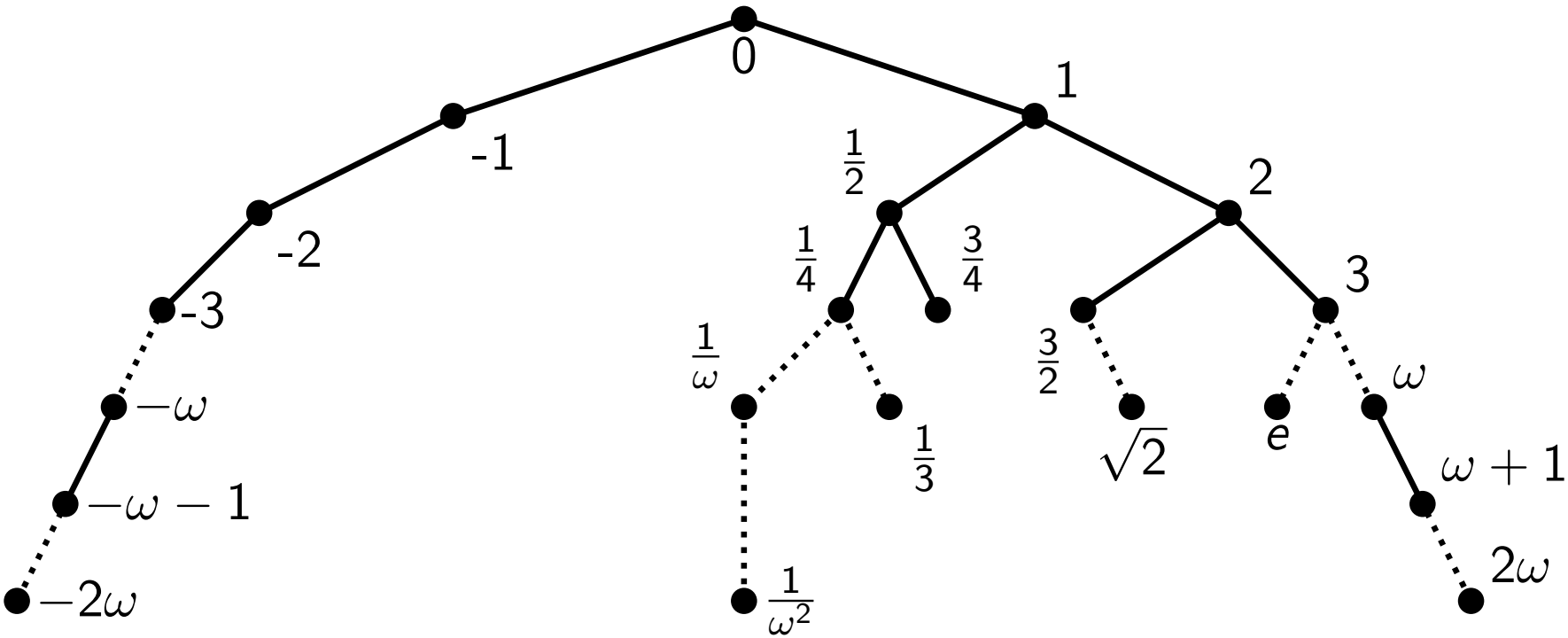
Also define

$$\{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} = \frac{1}{\omega} = \epsilon$$

Simplicity Tree



Expanded Simplicity Tree



Some More Surreal Numbers

If

$$\{1, 2, 3, ...|\} = \omega$$

then

$$\{\omega + 1, \omega + 2, \omega + 3, ...|\} = 2\omega$$

Some More Surreal Numbers

And if

$$\{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} = \frac{1}{\omega} = \epsilon$$

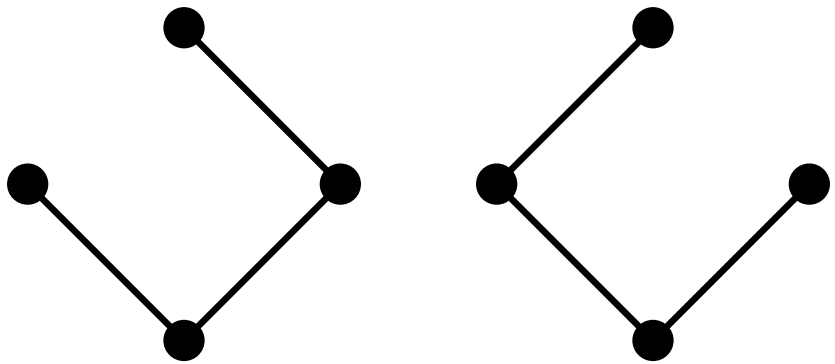
then

$$\{0|1, \frac{1}{2\omega}, \frac{1}{4\omega}, \frac{1}{8\omega}, \ldots\} = \frac{1}{\omega^2} = \epsilon^2$$

Let's Build Games from Numbers

Graph rules:

1. Left can only move northwest
2. Right can only move northeast
3. First player unable to move loses

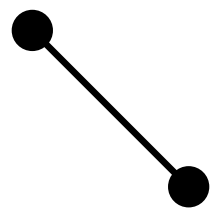


Let's Build Games from Numbers

$$\{|\} = 0 \implies$$

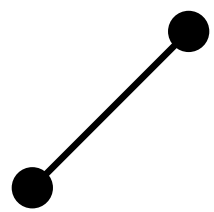
Let's Build Games from Numbers

$$\{0|\} = 1 \implies$$



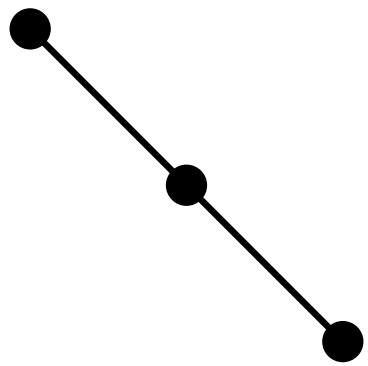
Let's Build Games from Numbers

$$\{|0\} = -1 \implies$$



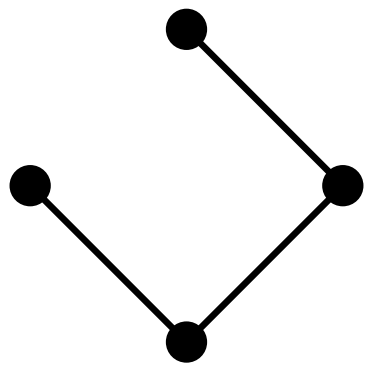
Let's Build Games from Numbers

$$\{1|\} = 2 \implies$$



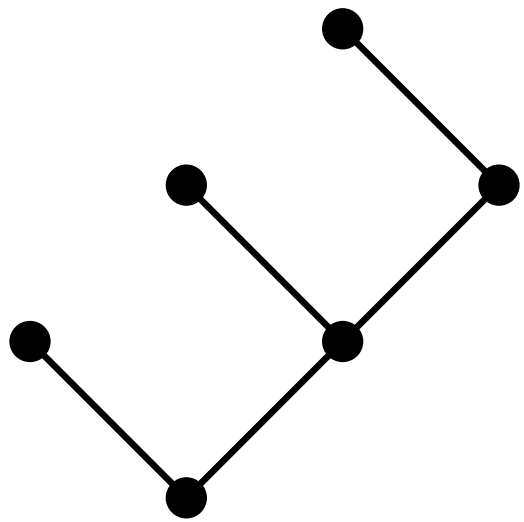
Let's Build Numbers from Games

$$\{0|1\} = \frac{1}{2} \Longrightarrow$$



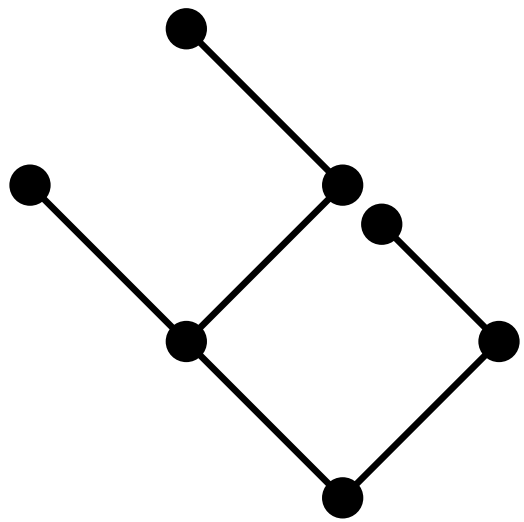
Let's Build Numbers from Games

$$\{0|\frac{1}{2}\} = \frac{1}{4} \implies$$



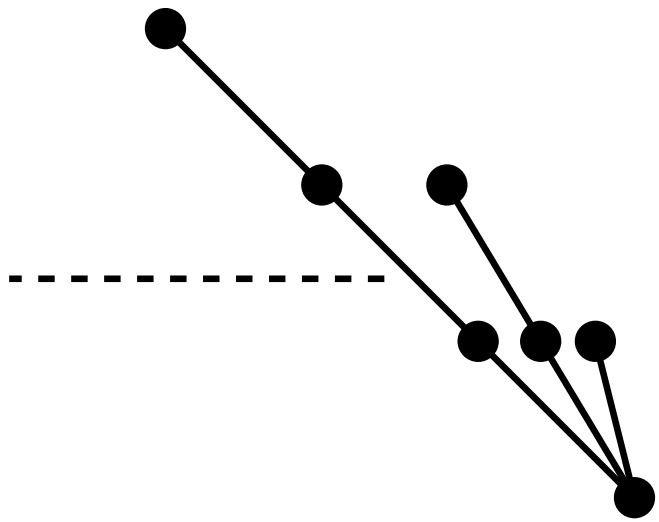
Let's Build Numbers from Games

$$\{\frac{1}{2}|1\} = \frac{3}{4} \implies$$



Surreal Graphs

$$\{0, 1, 2, \ldots | \} = \omega \implies$$



Bridging Numbers and Games

Definitions

If $G > 0$, Left will win.

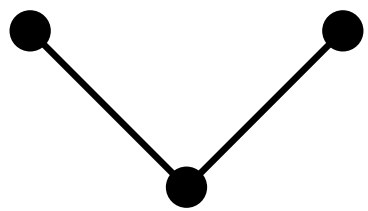
If $G < 0$, Right will win.

If $G = 0$, second player will win.

If $G || 0$, first player will win.

Other Games

$$\{0|0\} = * \implies$$



Incomparable with zero

We define a fuzzy game to be any game $\{x^L|x^R\}$ where:

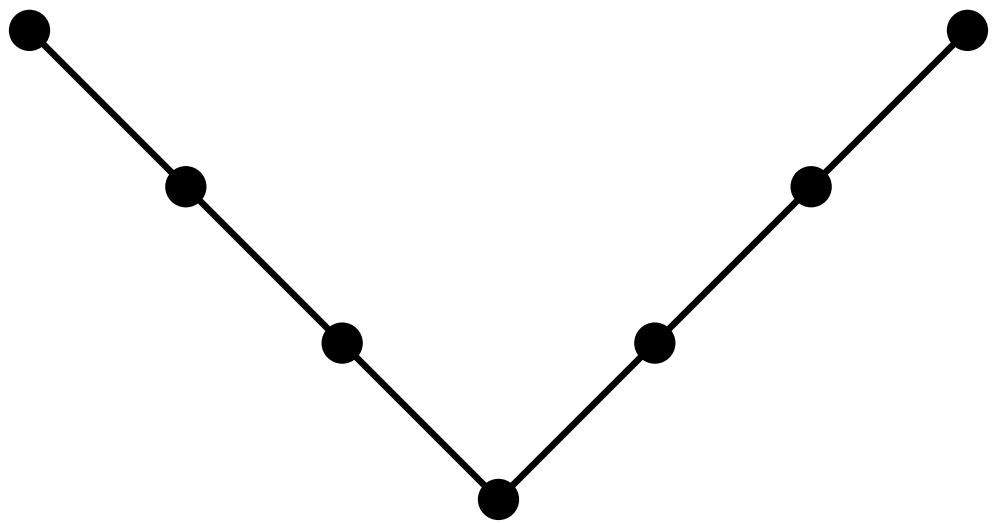
one $x^L >$ any x^R or

greatest $x^L =$ least $x^R = 0$

Other Games

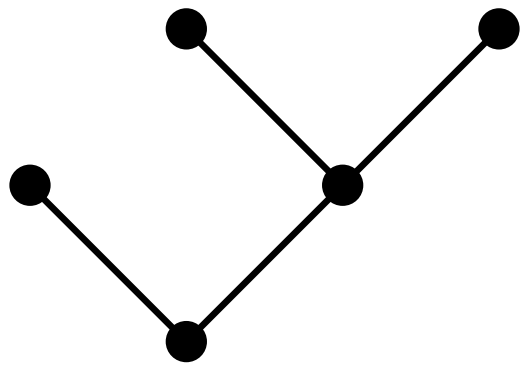
$$\{2|-2\}$$

is also fuzzy $\Longrightarrow$



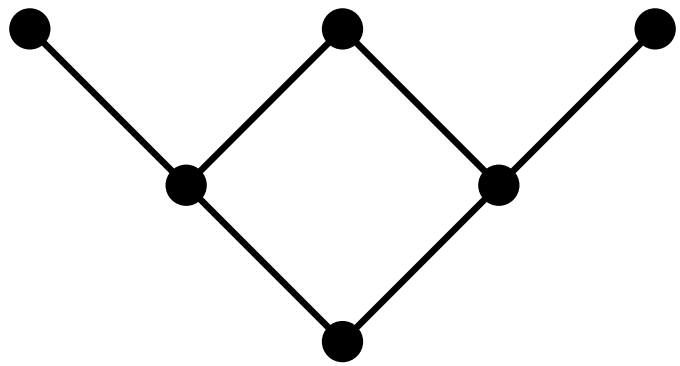
Let's Build Games from Games

$$\{0|*\} = \uparrow \Longrightarrow$$

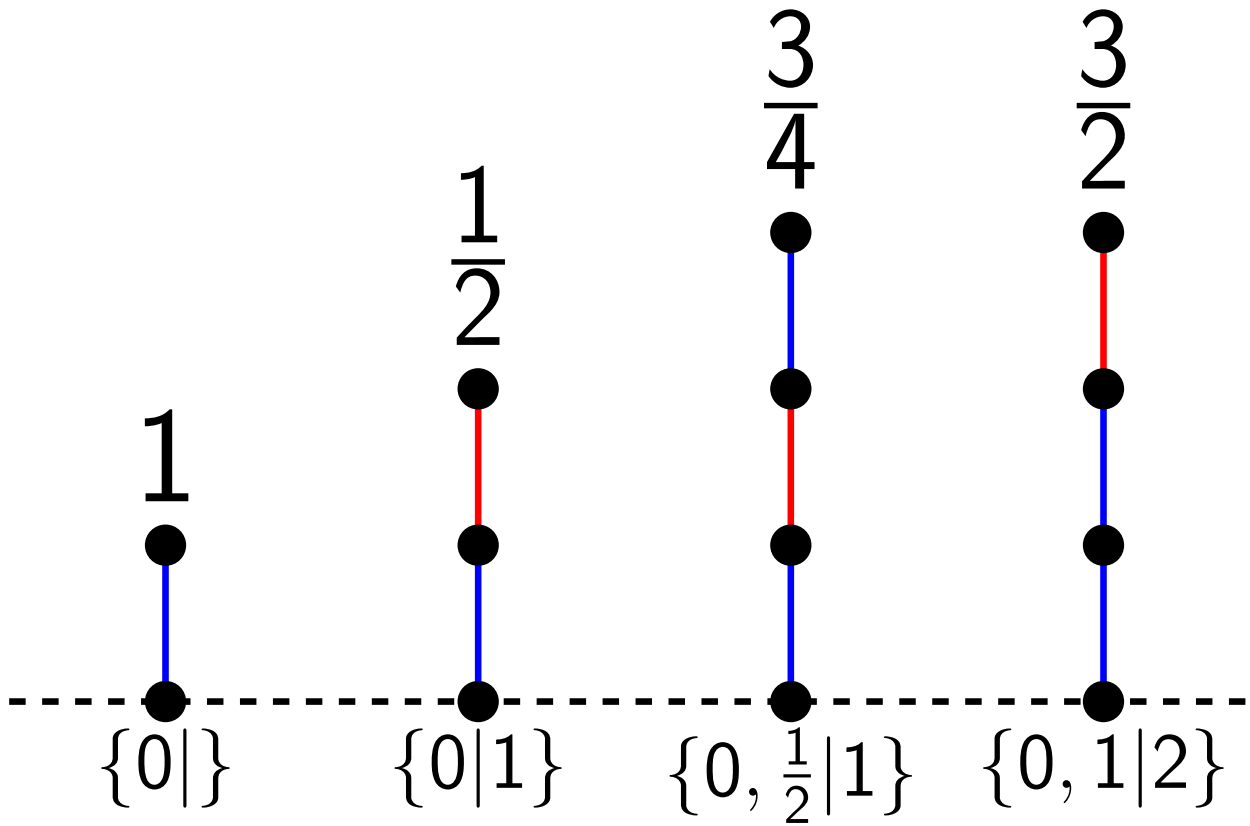


Games to Explain Real Numbers

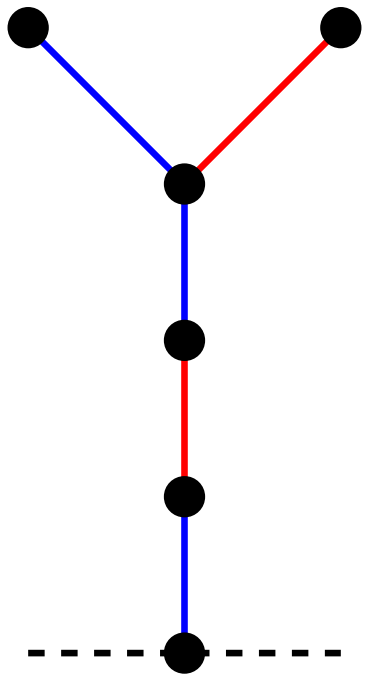
$$\{*|*\} = 0 \implies$$



Return to Hackenbush

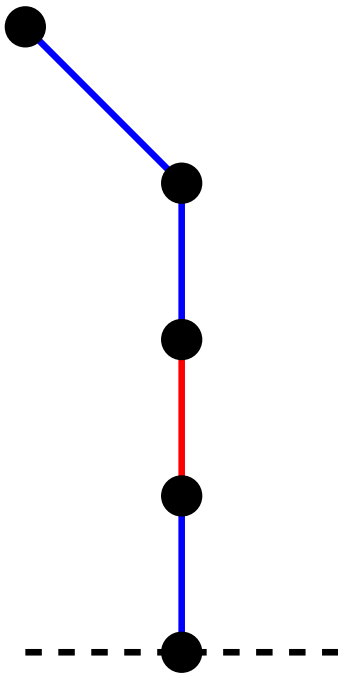


Recursive Hackenbush

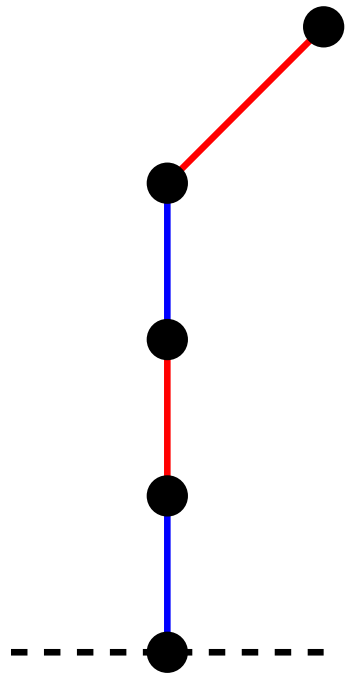


$$\{0, \frac{1}{2}, ?|1, ?\}$$

Recursive Hackenbush

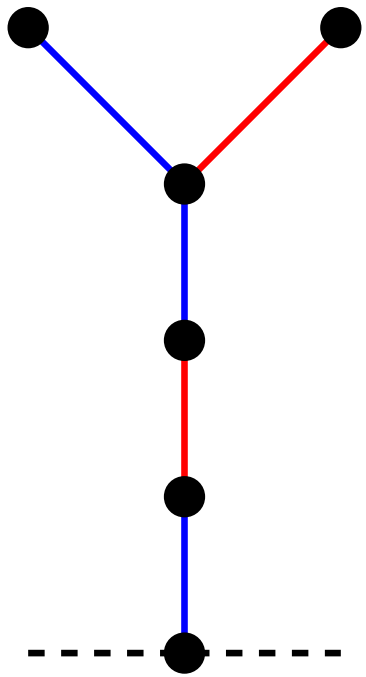


$\{0, \frac{1}{2}, \frac{3}{4} | 1\} = \frac{7}{8}$



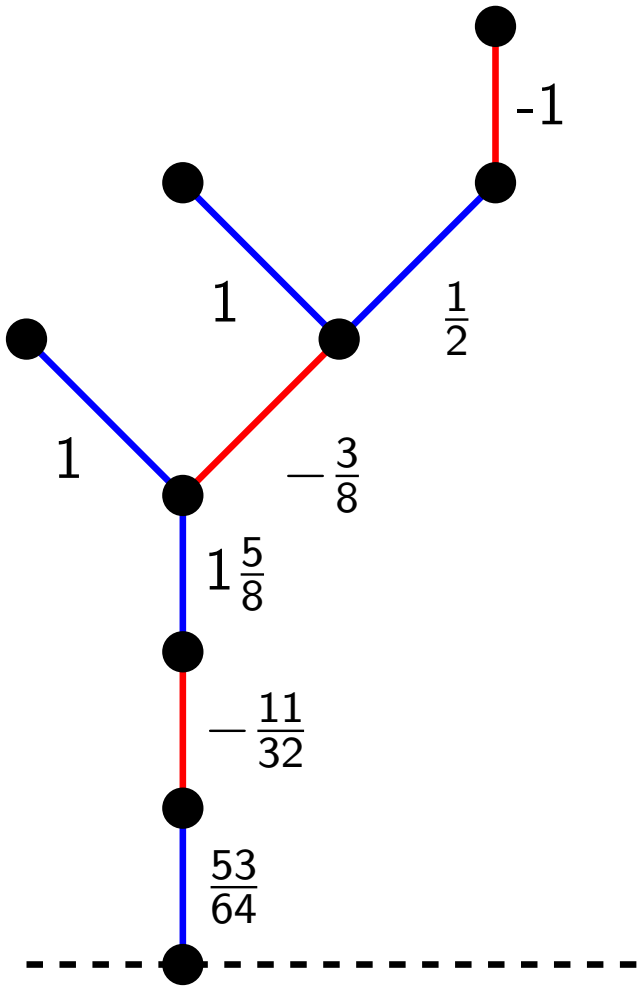
$\{0, \frac{1}{2} | 1, \frac{3}{4}\} = \frac{5}{8}$

Recursive Hackenbush

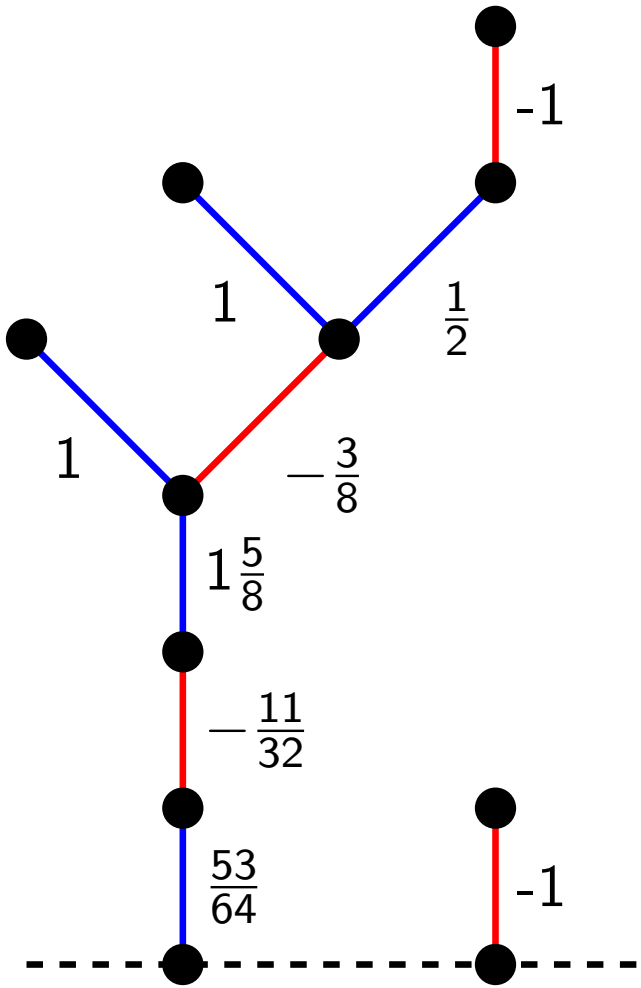


$$\{0, \frac{1}{2}, \frac{5}{8} | 1, \frac{7}{8}\} = \frac{3}{4}$$

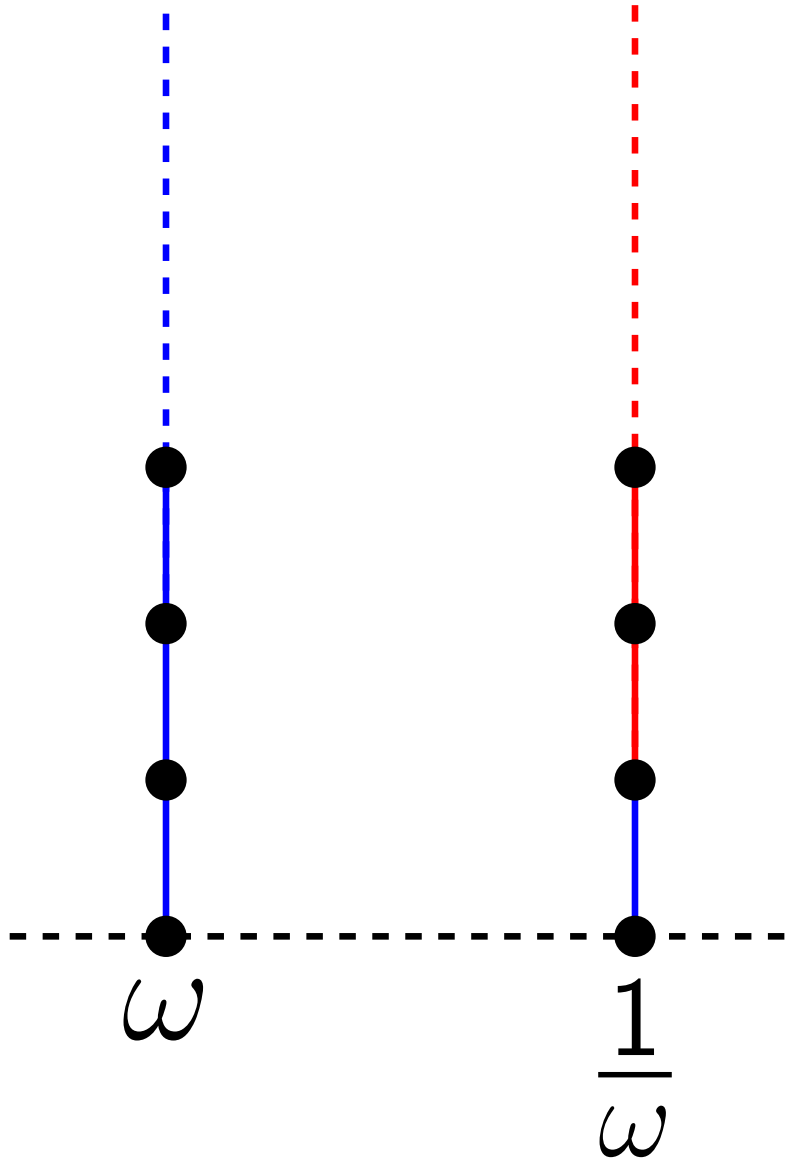
Return to Hackenbush



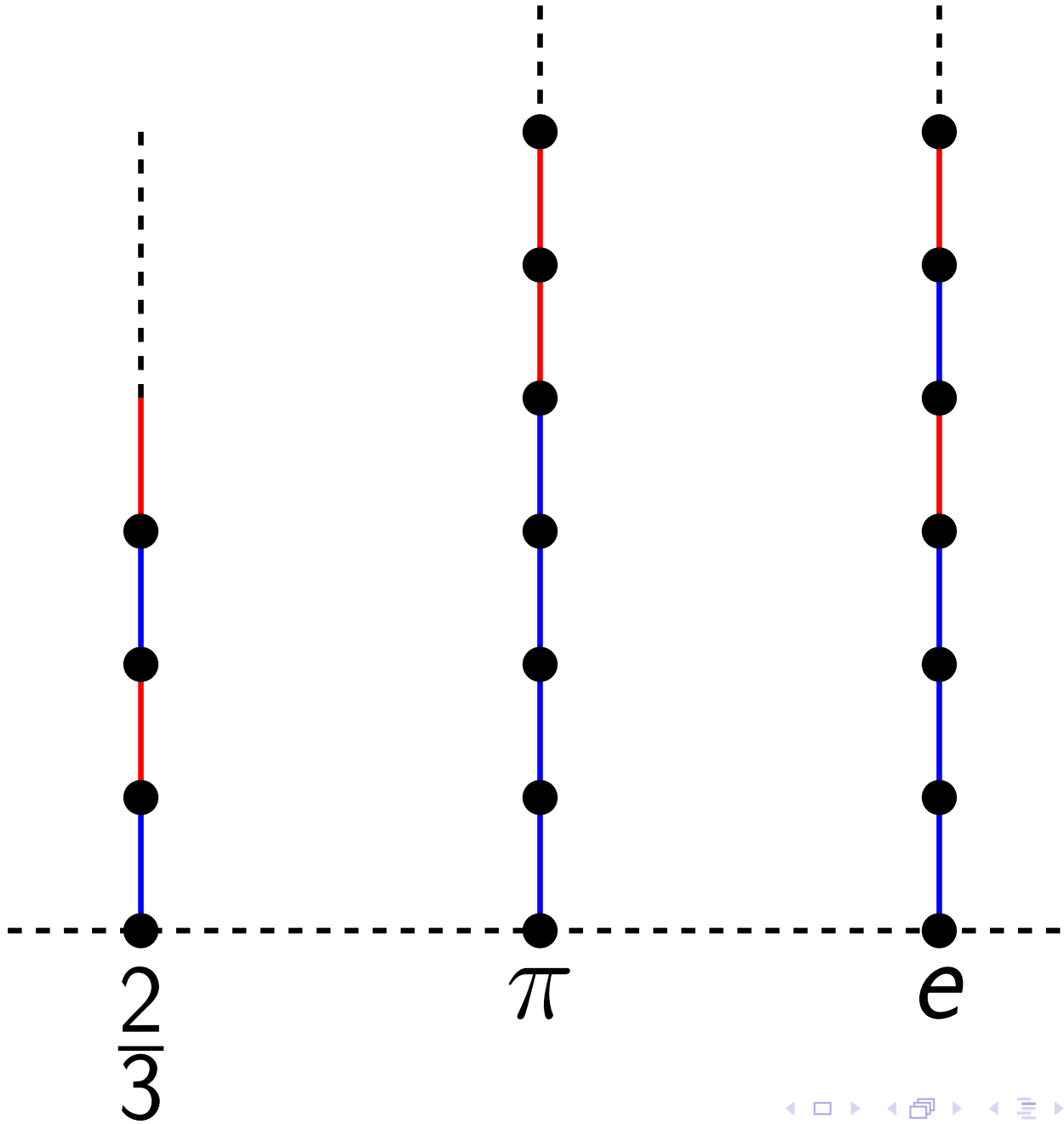
Return to Hackenbush

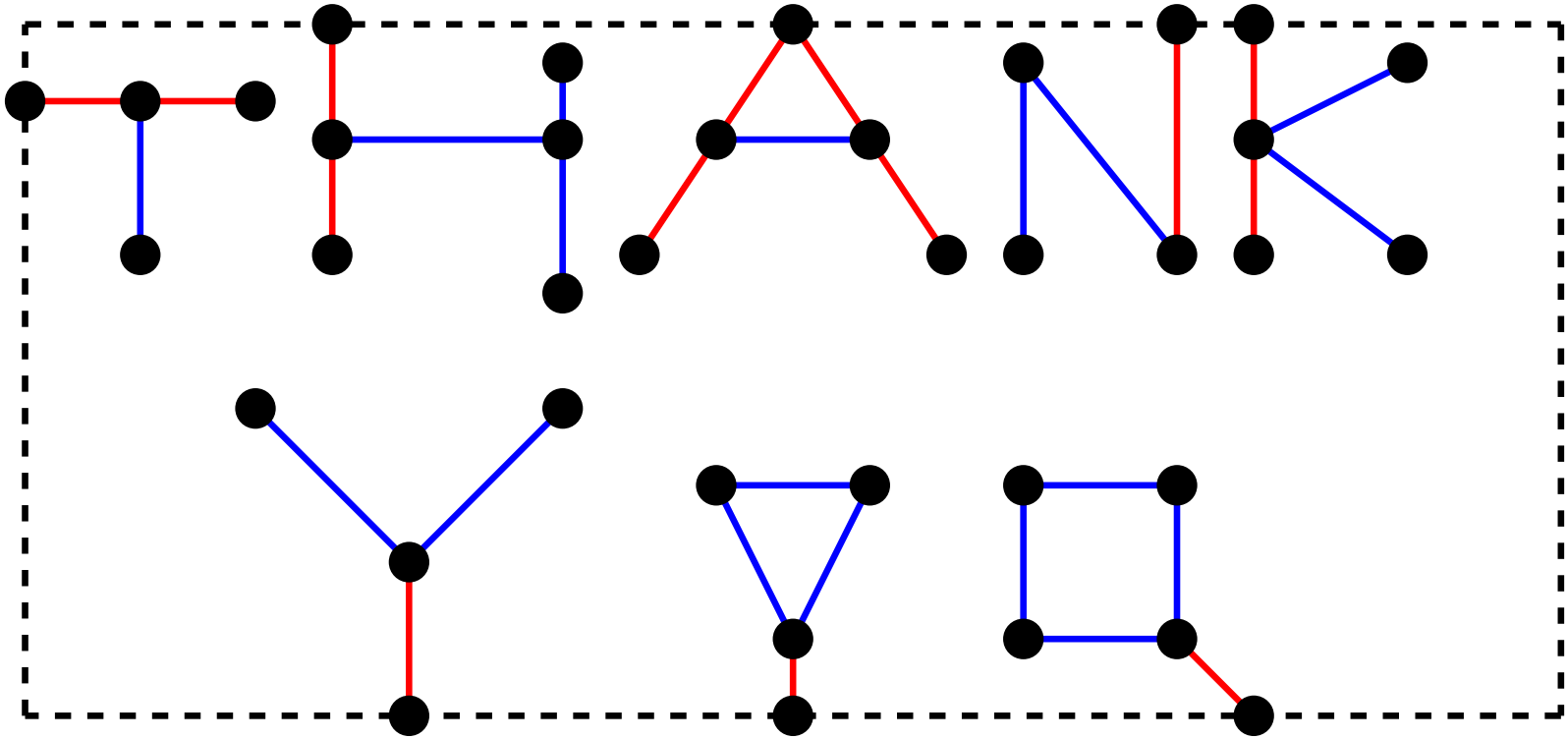


Surreal Hackenbush



Surreal Hackenbush





Work Cited

J. H. Conway. *On Numbers And Games*. A. K. Peters, Ltd, Natick, MA, 2001.