A Brief Sketch of the Prime Number Theorem

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1 Introduction

$\pi(x)$ = Number of primes less than or equal to X

The Theorem:

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1 \tag{1}$$

That is, $\frac{x}{log(x)}$ approximates $\pi(x)$ as x goes to ∞

This is called being asymptotic

2 Some Useful Functions (Mangolt, Mobius)

$$\Lambda(n) = \begin{matrix} \text{log(p) if n = p}^k \text{ for a prime } p \text{ and integer } k \leq 1 \\ \text{0 otherwise} \end{matrix} \tag{2}$$

$$\Psi(x) = \sum_{n\leq x} \Lambda(n) \tag{3}$$

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if n has a squared prime facto} \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors} \end{cases}$$

3 Selberg's Asymptotic Formula

$$\Psi(x) + \sum_{n\leq x} \Lambda(n)\Psi(\frac{x}{n}) = 2x\log(x) + O(x) \tag{4}$$

4 Outline of a Helpful Theorem

Theorem: For a function F defined on $(0, \infty)$, let

$$G(x) = \log x \sum_{n \leq x} (\frac{x}{n}) \tag{5}$$

Then

$$F(x) \log x + \sum_{n \leq x} F(\frac{x}{n}) \Lambda(n) = \sum_{d \leq x} \mu(d) G(\frac{x}{d}) \tag{6}$$

Important steps in the proof:

Rewrite as sum:

$$F(x) = \sum_{n \leq x} F(\frac{x}{n}) \log \frac{x}{n} \sum_{d|n} \mu(d) \tag{7}$$

We have another theorem which says that

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} \tag{8}$$

So we write

$$\sum_{n \leq x} F(\frac{x}{n}) \Lambda(n) = \sum_{n \leq x} F(\frac{x}{n}) \sum_{d|n} \mu(d) \log \frac{n}{d} \tag{9}$$

After simplifying

$$F(x) \log(x) + \sum_{n \leq x} F(\frac{x}{n}) \Lambda(n) \tag{10}$$

we get

$$\sum_{d \leq x} \mu(d) G(\frac{x}{d}) \tag{11}$$

5 Outline of Derivation of Selberg's Formula

$$\Psi(x) + \sum_{n \leq x} \Lambda(n) \Psi(\frac{x}{n}) = 2x \log(x) + O(x) \tag{12}$$

Apply the previous theorem to $F_1(x) = \Psi(x)$ and $F_2(x) = x - C - 1$
C is Euler's constant $\equiv .577$
Corresponding to F_1 is

$$G_1(x) = \log x \sum_{n \leq x} \Psi(\frac{x}{n}) = x \log^2 x - x \log x + O(\log^2 x) \tag{13}$$

The second equality comes from the theorem

$$\sum_{n \leq x} \Psi(\frac{x}{n}) = x \log x - x + O(\log x) \tag{14}$$

Corresponding to F_2 is

$$G_2(x) = \log x \sum_{n \leq x} F_2(\frac{x}{n}) = \log x \sum_{n \leq x} (\frac{x}{n} - C - 1) \tag{15}$$

After simplification, we get

$$= x \log x^2 - x \log x + O(\log x) \tag{16}$$

Subtracting $G_1(x)$ and $G_2(x)$ gives

$$G_1(x) - G_2(x) = O(\log^2 x) \tag{17}$$

It is more convient to use the easier

$$G_1(x) - G_2(x) = O(\sqrt{x}) \tag{18}$$

We apply the theorem to F_1 and F_2. Simplification yields

$$(\Psi(x) - (x - C - 1)) \log x + \sum_{n \leq x} (\Psi(\frac{x}{n}) - (\frac{x}{n} - C - 1))\Lambda(n) = O(x) \tag{19}$$

There is another theorem which says that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1) \tag{20}$$

Rearranging terms and using this theorem gives us

$$\Psi(x) \log x + \sum_{n \leq x} (\frac{x}{n})\Lambda(n) = (x - C - 1) \log x + \sum_{n \leq x} (\frac{x}{n} - C - 1)\Lambda(n) + O(x) = 2x \log x + O(x) \tag{21}$$

6 Finishing up

$$\sigma(n) = e^{-x}\Psi(e^x) - 1 \tag{22}$$

It can be shown that the PNT is equivalent to

$$\lim_{x->\infty} \sigma(x) = 0 \tag{23}$$

So if we let

$$C = \lim_{x->\infty} |\sigma(x)| \tag{24}$$

We can show that $C = 0$ to prove the PNT.
We know that

$$|\sigma(x)| \leq C + g(x) \tag{25}$$

where $g(x)$ is some function that goes to 0 as x goes to ∞.
Define D as

$$D = \lim_{x->\infty} (1/x) \int_0^x |\sigma(t)| \tag{26}$$

Selberg's formula implies that

$$|\sigma(x)|x^2 \leq O(x) + 2 \int_0^x \int_0^y |\sigma(u)| \tag{27}$$

This implies that $C \leq D$
If $C > 0$, then we can use Selberg's formula to show that

$$\frac{1}{x} \int_x^x |\sigma(t)| \leq C' + h(x) \tag{28}$$

where $0 < C' < C$ and $h(x)$ goes to 0 as x goes to ∞. If we do take $\lim_{x->\infty}$ of this function, we find that
$D \leq C' < C$
so C cannot be > 0, so C must $= 0$. As mentioned earlier, this is equivalent to showing that the PNT is true.