Solution to Calculus Placement Exam Practice Problems

1. If $\left(\frac{1}{2}\right)^{\frac{x}{15}} = \frac{1}{3}$, find the value of x.

   Take ln( ) on both sides:

$$\begin{aligned} \frac{x}{15}\ln(1/2) &= \ln(1/3) \\ x &= \frac{15\ln(1/3)}{\ln(1/2)} \text{ or } x = \frac{15\ln(3)}{\ln(2)} \end{aligned}$$

2. If $y = \frac{e^x}{1+e^x}$, solve for x.

   Multiply by $(1+e^x)$ on both sides:

$$\begin{aligned} (1+e^x)y &= e^x \\ y &= e^x(1-y) \\ \frac{y}{1-y} &= e^x \\ x &= \ln\left(\frac{y}{1-y}\right) \end{aligned}$$

3. If $\log_5 x + \log_5 9x = 2$, solve for x.

$$\begin{aligned} \log_5(x \cdot 9x) &= 2 \\ \log_5(9x^2) &= 2 \\ 9x^2 &= 5^2 \\ x &= \frac{5}{3} \quad (x = -\frac{5}{3} \text{ is not in the domain.}) \end{aligned}$$

4. Evaluate $\lim_{x\to 4} \frac{x^2-5x+4}{x^2-4x}$.

$$\lim_{x\to 4} \frac{(x-1)(x-4)}{x(x-4)} = \lim_{x\to 4} \frac{(x-1)}{x} = \frac{3}{4}$$

5. Evaluate $\lim_{x\to\infty} \sqrt{\frac{x+3x^2}{x^2+1}}$.

$$\lim_{x\to\infty} \sqrt{\frac{(x+3x^2)(1/x^2)}{(x^2+1)(1/x^2)}} = \lim_{x\to\infty} \sqrt{\frac{1/x+3}{1+1/x^2}} = \sqrt{3}$$

6. Evaluate $\lim_{x\to\pi/2} \sin\left(\frac{x}{3} + \cos x\right)$.

$$\sin\left(\frac{\pi}{6} + \cos(\pi/2)\right) = \sin\left(\frac{\pi}{6} + 0\right) = \frac{1}{2}$$

7. Find the equation of the line tangent to the graph of $y = \ln x$ at $x = e^3$.

$y'(x) = 1/x$, the slope of the tangent line at $x = e^3$ is $y'(e^3) = 1/e^3$.
The y value at $x = e^3$ is $\ln(e^3) = 3$.
The line equation is

$$(y-3) = \frac{1}{e^3}(x - e^3) \text{ or } y = \frac{x}{e^3} + 2$$

8. Let $f(x) = \sqrt{x^2+1}$, find $f'(x)$.

Chain rule: $f'(x) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \dfrac{x}{\sqrt{x^2+1}}$

9. Let $f(x) = \sin^2 x$, find $f'(\pi/4)$.

Chain rule: $f'(x) = 2\sin x\cos x$, $f'(\pi/4) = 2(\sqrt{2}/2)(\sqrt{2}/2) = 1$

10. Let $y = x^3\cos(2x)$, find $\dfrac{dy}{dx}$.

Product rule: $\dfrac{dy}{dx} = 3x^2\cos(2x) + x^3(-2\sin(2x)) = 3x^2\cos(2x) - 2x^3\sin(2x)$

11. Let $y = \dfrac{x^2+1}{x-1}$, find $\dfrac{dy}{dx}$.

Quotient rule: $\dfrac{dy}{dx} = \dfrac{(2x)(x-1) - (1)(x^2+1)}{(x-1)^2} = \dfrac{x^2-2x-1}{(x-1)^2}$

12. Let $f(x) = \sqrt{x}$, evaluate $\dfrac{f(x+h) - f(x)}{h}$.

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

13. Given the implicit equation $\sin(y) = xy + x^2$, find $\dfrac{dy}{dx}$.

$$\begin{aligned}(\cos y)y' &= 1y + xy' + 2x \\ (\cos y - x)y' &= y + 2x \\ y' &= \frac{y+2x}{\cos y - x}\end{aligned}$$

14. Let $f(x) = x + e^{-2x}$, find the interval on which $f(x)$ is increasing.

$f'(x) = 1 - 2e^{-2x} = 0$, critical point at $x = \frac{\ln(1/2)}{-2} = \frac{\ln(2)}{2}$. $f'(x) > 0$ when $x > \frac{\ln(2)}{2}$, $f'(x) < 0$ when $x < \frac{\ln(2)}{2}$, so $f(x)$ is increasing on $\left(\frac{\ln(2)}{2}, \infty\right)$.

15. Let $f(x) = x^2 + \ln(3x)$, find the inflection point of $f(x)$.

$f'(x) = 2x + \dfrac{1}{x}$, $f''(x) = 2 - \dfrac{1}{x^2} = \dfrac{2x^2-1}{x^2}$, $f''(x) = 0$ and $f''(x)$ changes sign at $x = \dfrac{1}{\sqrt{2}}$. ($x = -\dfrac{1}{\sqrt{2}}$ is not in the domain.)

16. Evaluate $\lim\limits_{\theta\to 0} \dfrac{\sin(2\theta)}{3\theta}$.

L'Hospital's rule: $\lim\limits_{\theta\to 0} \dfrac{2\cos(2\theta)}{3} = \dfrac{2}{3}$.

17. Let $f(x) = x^3 - 3x^2 - 9x + 1$, find the x value at which $f(x)$ has a local minimum.

$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0$, critical point at $x = -1, 3$. $f'(x) > 0$ on $(-\infty, -1) \cup (3, \infty)$, $f'(x) < 0$ on $(-1, 3)$. $f(x)$ has local maximum at $x = -1$, local minimum at $x = 3$.

18. Evaluate the indefinite integral $\displaystyle\int (1 + \tan x)^3 \sec^2 x \, dx$.

u–substitution: Let $u = 1 + \tan x$, $du = \sec^2 x dx$,

$\displaystyle\int u^3 du = \frac{u^4}{4} + C = \frac{(1+\tan x)^4}{4} + C.$

19. Evaluate the indefinite integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

u–substitution: Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$, $\int e^u 2du = 2e^u + C = 2e^{\sqrt{x}} + C$.

20. Evaluate the indefinite integral $\int \frac{x^2}{1+x^3}\, dx$.

u–substitution: Let $u = 1+x^3$, $du = 3x^2dx$, $\int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|1+x^3|+C$.

21. For what value of b is the definite integral $\int_1^b \frac{3}{x^2}\, dx = 1$.

$\int_1^b \frac{3}{x^2}dx = -\frac{3}{x}\Big|_1^b = -3\left(\frac{1}{b} - \frac{1}{1}\right) = 1$, $b = \frac{3}{2}$.

22. If $F'(x) = 2x^3 + \frac{1}{x}$ and $F(1) = 2$, find $F(x)$.

Antiderivative $F(x) = 2\frac{x^4}{4} + \ln|x| + C$, $F(1) = \frac{1}{2} + 0 + C = 2$, $C = \frac{3}{2}$, so $F(x) = \frac{x^4}{2} + \ln|x| + \frac{3}{2}$

23. Find the area bounded between the curves $y = x^2 - 1$ and $y = 1 - x$.

Intersection point: $x^2 - 1 = 1 - x$, $x^2 + x - 2 = (x+2)(x-1) = 0$, $x = -2, 1$.
Area $= \int_{-2}^{1} [(1-x) - (x^2-1)]dx = \int_{-2}^{1}(-x^2 - x + 2)dx = \frac{-x^3}{3} - \frac{x^2}{2} + 2x\Big|_{-2}^{1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = \frac{9}{2}$

24. Set up the integral to find the volume of the solid obtained by revolving the region enclosed by the curves $y = 2x^2 + 1$ and $y = x^2 + 5$ about the x–axis.

Intersection point: $2x^2 + 1 = x^2 + 5$, $x^2 - 4 = 0$, $x = -2, 2$.
Volume $= \int_{-2}^{2} \pi[(x^2+5)^2 - (2x^2+1)^2]dx = \pi\int_{-2}^{2}[x^4 + 10x^2 + 25 - (4x^4 + 4x^2 + 1)]dx = \pi\int_{-2}^{2}(-3x^4 + 6x^2 + 24)dx$